Matrix models

why matrices

What are matrix models

How are matrix models solved

why:

(a) Modelling chaotic hamiltonians

(b) Random surfaces, QG2

(c) Feynman diagrams of Yang Mills

(d) String theory / D-branes

REASON (a)

Consider a very symmetric system: 2D harmonic oscillator

$$H = \frac{\vec{p}^{\,2}}{2} + \frac{\vec{x}^{\,2}}{2}$$

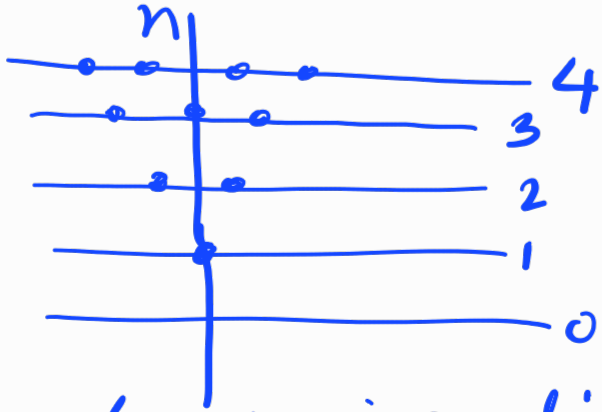
$$E_{n_1, n_2} = (n_1 + n_2 + 1)\hbar \equiv n\hbar$$

$n = 1 \quad (n_1, n_2) = (0, 0)$

$n = 2 \quad (0,1), (1,0)$

$n = 3 \quad 02, 11, 20$

Symmetry $\Rightarrow$ degeneracy



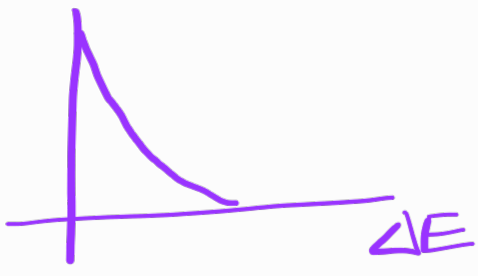
level spacing distribution
up to energy $n=4$

Call these states (in binary order)
$E_1 \leq E_2 \leq E_3 \leq \cdots \leq E_{10}$
E_{00} E_{10} E_{01}

$\Delta E \equiv E_i - E_{i+1}$

$\Delta E = 0 \Rightarrow 1+2+3 = 6$
$\Delta E = 1 \Rightarrow 1+1+1 = 3$
$\Delta E = 2 \Rightarrow 1+1 = 2$

Many more degenerate states than otherwise



By contrast, consider a Hamiltonian with all degeneracies broken
$\Rightarrow$ no $\Delta E = 0$ (i.e. $E_i \neq E_{i+1} \; \forall i$)

(e.g. $H = (n_1+\frac{1}{2})\hbar\omega_1 + (n_2+\frac{1}{2})\hbar\omega_2$

$\omega_1 < \omega_2$ irrationally related

$$E_{n_1,n_2} \overset{?}{=} E_{m_1,m_2}$$

$$(m_1 - n_1)\omega_1 = (n_2 - m_2)\omega_2$$

not possible since $\frac{\omega_1}{\omega_2} \neq$ rational)

Consider a many-particle hamiltonian like that of a heavy nucleus

Wigner $\Rightarrow$ the hamiltonian is so complicated that detailed structure of the spectrum is very difficult to know and in some sense, may not be important.

(cf. molecules of air in a room $\Rightarrow$ very difficult to solve the motion of individual molecules, but some average characteristics of motion are universal ⟺ thermodynamics)

Wigner proposed that a very complex hamiltonian e.g. that of a heavy nucleus is fairly well described by a random hamiltonian.

More precisely, the matrix elements H_{ij} the hamiltonian should be regarded as independent random variables, defined by some probability density

"$\prod_{ij} p_{ij}(\{H_{ij}\})\, dH_{ij}$"

Subject to some very general considerations

(a) whether the system is time reversal (T) invariant $\Rightarrow$ $H_{ij} \in \mathbb{R}$ $H_{ij} = H_{ji}$ $(T^2 = 1)$

i.e. H_{ij} = real symmetric

$T\psi = \psi^*$ $\Rightarrow T^2 = 1$

$\left[\text{e.g } H_{\vec{x}\vec{y}} = \left(-\frac{1}{2m}\partial^2_{\vec{x}} + V(\vec{x})\right)\delta(\vec{x}-\vec{y}) \right]$

(or $H_{ij} = H^{(0)}_{ij}\, \mathbb{1}_2 + \sum_{\lambda=1,2,3} i H^{(\lambda)}_{ij} \sigma_\lambda$

where $H^{(0)}_{ij} = H^{(0)}_{ji} \in \mathbb{R}$, $H^{(\lambda)}_{ij} = -H^{(\lambda)}_{ji} \in \mathbb{R}$ $\longleftarrow$ real quaternion self dual

for $T^2 = -\mathbb{1}$)

[e.g. $H = -\frac{1}{2m}\partial^2_{\vec{x}} + g\,\vec{\sigma}.\vec{L}$

$\vec{\sigma}.\vec{L}$ = spin-orbit coupling

$= \sigma_1 (-i\, y\partial_z + i z \partial_y)\, \delta(\vec{x}-\vec{x}')$
+ cycl. perm.

$= i\sigma_1 (z\partial_y - y\partial_z)\, \delta(y-y')\delta(z-z')\delta(x-x')$
+ c.p.

$T\psi = i\sigma_2\psi^*$ $T^2 = -\mathbb{1}$

Each term of form $\partial_y \delta(y-y')$ is a real antisymmetric matrix]

Or whether H is not time-reversal invariant

$\Rightarrow$ H is hermitian $H = H^R_{ij} + i H^I_{ij}$

$H^R_{ij} = + H^R_{ji}$ $\quad H^I_{ij} = - H^I_{ji}$

e.g. $H_{\vec{x},\vec{y}} = -\frac{1}{2m}(-i\partial_{\vec{x}} - A_{\vec{x}})^2\, \delta(\vec{x}-\vec{y}) = H^R_{\vec{x},\vec{y}} + i H^I_{\vec{x},\vec{y}}$

$H^*_{\vec{x},\vec{y}} = H^R_{\vec{x},\vec{y}} - i H^I_{\vec{x},\vec{y}} = H^R_{\vec{y},\vec{x}} + i H^I_{\vec{y},\vec{x}}$

In the first case of real symmetric hamiltonian the randomness ansatz translates to

$$\text{prob} = \prod_i dH_{ii}\, P_i(H_{ii}) \prod_{i<j} dH_{ij}\, P_{ij}(H_{ij})$$

For the hermitian case

$$\text{prob} = \prod_i dH_{ii}\, P_i(H_{ii}) \prod_{i<j} dH^R_{ij}\, P^R_{ij}(H^R_{ij})$$

$$\prod_{i<j} dH^I_{ij}\, P^I_{ij}(H^I_{ij})$$

For the real self-dual (symplectic case)

$$\text{prob} = \prod_{i<j} dH^{(0)}_{ij}\, P^{(0)}_{ij}(H^{(0)}_{ij}) \prod_{\substack{\lambda \\ i<j}} dH^{(\lambda)}_{ij}\, P^{(\lambda)}_{ij}(H^{(\lambda)}_{ij})$$

(b) The second condition is that the prob distribution should be basis independent. A basis change is reflected as follows

real sym: $H \to O H O^T$

O = real orthogonal

hermitian: $H \to U H U^\dagger$

U = unitary

real self-dual quaternionic

$H \to S^R H S$ $\quad S$ = symplectic

$S^R \equiv (i\sigma_2) S (i\sigma_2)^{-1}$ = dual of S

<u>Dyson</u>: The probab. densities are given by

(1) $\text{prob} = \prod_i dH_{ii} \prod_{i<j} dH_{ij}\, e^{-(a\,\mathrm{Tr}H^2 + b\,\mathrm{Tr}H + c)}$

for real symmetric

(2) $\text{prob} = \prod_i dH_{ii} \prod_{i<j} dH^R_{ij}\, dH^I_{ij}\, e^{-(a\,\mathrm{Tr}H^2 + b\,\mathrm{Tr}H + c)}$

for hermitian

(3) $\text{prob} = \prod_{i\leq j} dH^{(0)}_{ij} \prod_{i<j} dH^{(1)}_{ij}\, dH^{(2)}_{ij}\, dH^{(3)}_{ij}\, e^{-(a\,\mathrm{Tr}H^2 + b\,\mathrm{Tr}H + c)}$

for real selfdual quaternionic case

(1) $\longrightarrow$ GOE

(2) $\longrightarrow$ GUE

(3) $\longrightarrow$ GSE

Proof: 2×2 GOE $\quad H = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

$\text{prob} = da\, db\, dc\;\; p_1(a)\, p_2(b)\, p_3(c)$

under $\mathcal{O} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \mathbb{1} + \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$H' = H + \theta \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right]$ $\qquad \begin{pmatrix} b & c \\ -a & -b \end{pmatrix}$

$= H + \theta \begin{pmatrix} 2b & c-a \\ c-a & -2b \end{pmatrix}$ $\qquad -\begin{pmatrix} -b & a \\ -c & b \end{pmatrix}$

$\delta a = 2\theta b \quad \delta b = \theta(c-a) \quad \delta c = -2\theta b$

Jacobian

$$J = \begin{pmatrix} 1 & 2\theta & 0 \\ -\theta & 1 & \theta \\ 0 & -2\theta & 1 \end{pmatrix} \xrightarrow{\det} 1 + O(\theta^2) \to 1$$

$\therefore$ we must have

$$p_1(a)\, p_2(b)\, p_3(c) = p_1(a')\, p_2(b')\, p_3(c')$$

$$\delta a\, \partial_a \ln p_1(a) + \delta b\, \partial_b \ln p_2(b) + \delta c\, \partial_c \ln p_3(c) = 0 \quad \text{—— (1)}$$

Independently, there is an orthogonal matrix $O = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ s.t. $O\begin{pmatrix} a & b \\ b & c \end{pmatrix}O^T = \begin{pmatrix} b & c \\ -a & -b \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$

$\Rightarrow \quad p_1 = p_3 \quad$ and $\quad p_2(b) = p_2(-b)$

Put $p_1 = p_3 = p$ in (1)

$$0 = 2\theta b\left(\partial_a \ln p(a) - \partial_c \ln p(c)\right) + \theta(c-a)\partial_b \ln p_2(b)$$

$a \pm c = a_\pm \qquad \partial_a = \partial_+ + \partial_- \,, \quad \partial_c = \partial_+ - \partial_-$

$$0 = \frac{2}{(c-a)}\left(\partial_a \ln p(a) - \partial_c \ln p(c)\right) + \frac{1}{b}\partial_b \ln p_2(b) = 0$$

Both terms must be a constant $(\alpha) + (-\alpha) = 0$

$$\frac{\partial_b \ln p_2(b)}{b} = -\alpha \qquad \ln p_2(b) = -\frac{\alpha b^2}{2} + \gamma$$

$$p_2(b) = \exp\left(-\frac{\alpha b^2}{2} + \gamma\right)$$

and $\frac{2}{(c-a)}\left(\partial_a \ln p(a) - \partial_c \ln p(c)\right) = \alpha$ —— (2)

Let $\ln p(a) = \sum_{n=1}^{\infty} b_n \frac{a^n}{n} + b_0 = b_0 + b_1 a + b_2 \frac{a^2}{2} + \cdots$

$\Rightarrow$ (2) $\Rightarrow$ $\frac{1}{c-a}\sum_{n=1}^{\infty} b_n\left(a^{n-1} - c^{n-1}\right) = \frac{\alpha}{2}$

$\Rightarrow$ $b_1 = b_1$ $\quad b_2 = b_2 \quad b_3 = b_4 = \cdots = 0$

and $\frac{1}{c-a} b_2 (a-c) = \frac{\alpha}{2}$

$\Rightarrow$ $b_2 = -\frac{\alpha}{2}$

$\therefore$ $\ln p(a) = b_0 - \frac{\alpha a^2}{4} + b_1 a$, $p(a) = e^{-\frac{\alpha a^2}{4} + b_1 a + b_0}$

$\therefore$ prob $=$ $da\, db\, dc \exp\left[-\frac{\alpha}{2}\left(\frac{a^2+c^2}{2} + b^2\right) + b_1(a+c) + b_0\right]$

C: const (to be chosen by $\int$prob $= 1$)

$$\mathrm{Tr}\Big(\underbrace{\begin{pmatrix} a & b \\ b & c\end{pmatrix}^2}_{\begin{pmatrix} a & b \\ b & c\end{pmatrix}\begin{pmatrix} a & b \\ b & c\end{pmatrix} = \begin{pmatrix} a^2+b^2 & * \\ * & b^2+c^2\end{pmatrix}}\Big) = \mathrm{Tr}\left(a^2 + c^2 + 2b^2\right)$$

$\therefore$ prob $= \underbrace{da\,db\,dc}_{dH} \times$ const $\times \exp\left[-\frac{1}{2\sigma^2}\mathrm{Tr}\, H^2 + b_1 \mathrm{Tr}\, H\right]$

$\Rightarrow$ independently distributed matrix elements + rotational invariance $\Rightarrow$ Gaussian !!

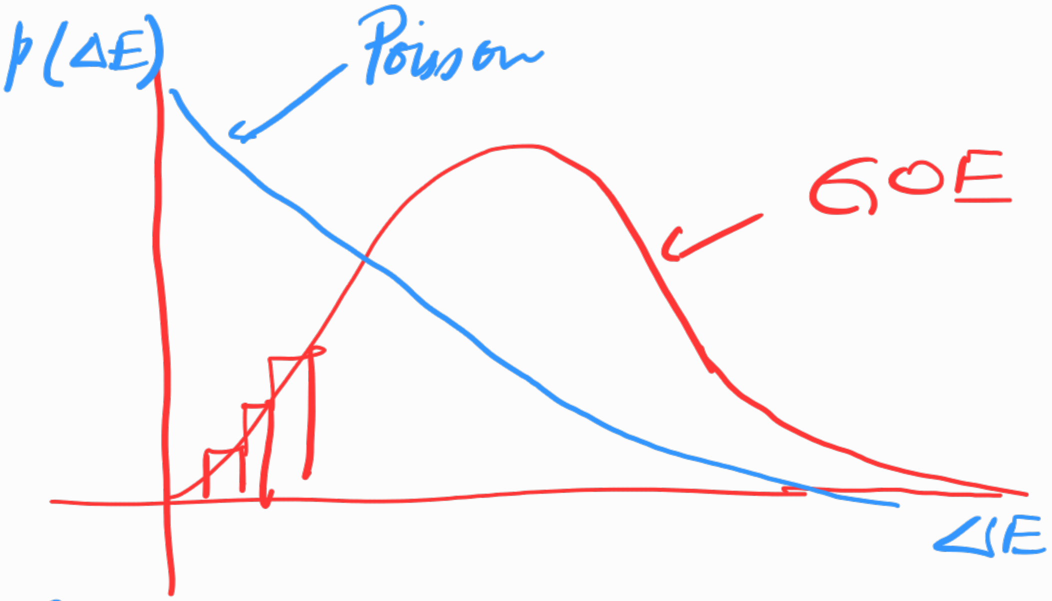
Does this ansatz work ?

$$E_{i+1} \;\text{———}\; \Big\} \text{ slow neutron}$$
$$E_i \;\text{———}$$

Look at <u>M. L. Mehta Random matrices</u>

<u>Brody et al RMΦ</u>

166 Erbium (108 level spacings)



Probab vanishes at $\Delta E = 0$ i.e. no degeneracy.

A computation:

2×2 GOE

$$\text{prob} = da\, db\, dc\, (\text{const})\, e^{-\frac{a^2+2b^2+c^2}{2\sigma^2}}$$

$$H = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = O \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} O^T$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\mathrm{Tr}\, H^2 = \lambda_1^2 + \lambda_2^2$$

$$da\, db\, dc = |\lambda_1 - \lambda_2|\, d\lambda_1 d\lambda_2\, d\theta$$

(Proof: $a = \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta$

$b = (\lambda_2 - \lambda_1) \sin\theta \cos\theta$

$c = \lambda_1 \sin^2\theta + \lambda_2 \cos^2\theta$

$J\left(\frac{a,b,c}{\lambda_1,\lambda_2,\theta}\right) = |\lambda_1 - \lambda_2|$]

Alternatively,
$da^2 + db^2 + dc^2$
$= d\lambda_1^2 + d\lambda_2^2$
$+ (\lambda_1 - \lambda_2)^2 d\theta^2$

$da\,db\,dc$
$= |\lambda_1 - \lambda_2| d\lambda_1 d\lambda_2 d\theta$

$\Rightarrow$ prob $= |\lambda_1 - \lambda_2|\, e^{-\frac{\lambda_1^2+\lambda_2^2}{2}}\, d\lambda_1 d\lambda_2\, d\theta$ $(\sigma = 1)$

Marginal prob. distribution of $x = \lambda_1 - \lambda_2 = \lambda_-$

$\lambda_\pm = \lambda_1 \pm \lambda_2$

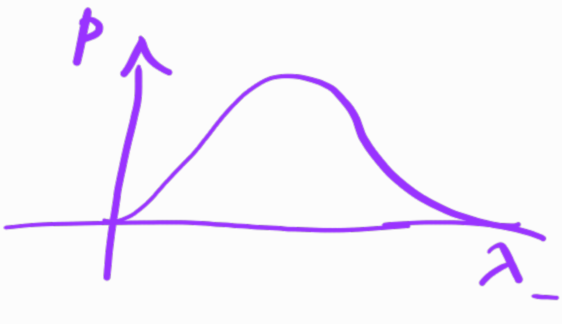
$d\lambda_1 d\lambda_2 = \#\, d\lambda_+ d\lambda_-$

$$\frac{\lambda_1^2+\lambda_2^2}{2} = \frac{\lambda_+^2 + \lambda_-^2}{4}$$

prob $= \#\, d\theta\, d\lambda_+ d\lambda_-\, e^{-\frac{\lambda_+^2+\lambda_-^2}{4}}\, |\lambda_-|$

marginal prob. of λ_-

$$\#\, d\lambda_-\, \underbrace{e^{-\frac{\lambda_-^2}{4}} |\lambda_-|}_{p(\lambda_-)}$$



Note the level repulsion $p(0) = 0$

More generally,

$N \times N$ matrix, $x \equiv \lambda_1 - \lambda_2$

$$p(x) = \# \, |x| \, (1 + O(x)) \exp\left(-\frac{x^2}{4}\right)$$

Hence there is a universal level spacing distribution near $x \to 0$

REASON (b)

Random surfaces

$$Z = \int dM \, e^{-\mathrm{Tr}\frac{M^2}{2g}}$$

GUE

$M = N \times N$ Herm. matrix

Consider

$$\frac{1}{N}\langle \mathrm{Tr}\, M^4 \rangle = \frac{1}{Z} \int dM \, \mathrm{Tr}\, M^4 \, e^{-S} \qquad (3)$$

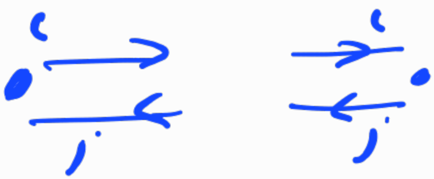
$$S = \frac{1}{2g} \mathrm{Tr}\, M^2$$

Consider this computation in terms of a Feynman diagram.

$$dM_{ij} \, dM^*_{ij} \exp\left(-\frac{1}{2g} M_{ij} M^*_{ij}\right)$$

$$\Rightarrow \langle M_{ij} M^*_{kl} \rangle = g \, \delta_{ik} \delta_{jl}$$

$$\Rightarrow \langle M_{ij} M_{lk} \rangle = g \, \delta_{ik} \delta_{jl}$$



continuity of double arrow