Lecture 2

The other reasons for considering matrix models

REASON (b)

Random surfaces

$$Z = \int dM \, e^{-\mathrm{Tr}\frac{M^2}{2g}}$$

GUE
$M = N\times N$ Herm. matrix

Consider

$$\frac{1}{N}\langle \mathrm{Tr}\, M^4 \rangle = \frac{1}{Z}\int dM \, \mathrm{Tr}\, M^4 \, e^{-S} \qquad (3)$$

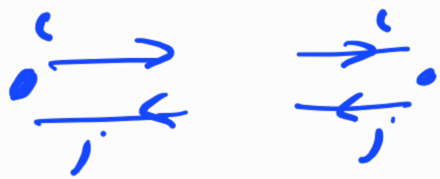
$$S = \frac{1}{2g}\mathrm{Tr}\, M^2$$

Consider this computation in terms of a Feynman diagram.

$$dM_{ij}\, dM^*_{ij} \exp\left(-\frac{1}{2g} M_{ij} M^*_{ij}\right)$$

$$\Rightarrow \langle M_{ij} M^*_{kl}\rangle = g\,\delta_{ik}\delta_{jl}$$

$$\Rightarrow \langle M_{ij} M_{lk}\rangle = g\,\delta_{ik}\delta_{jl}$$

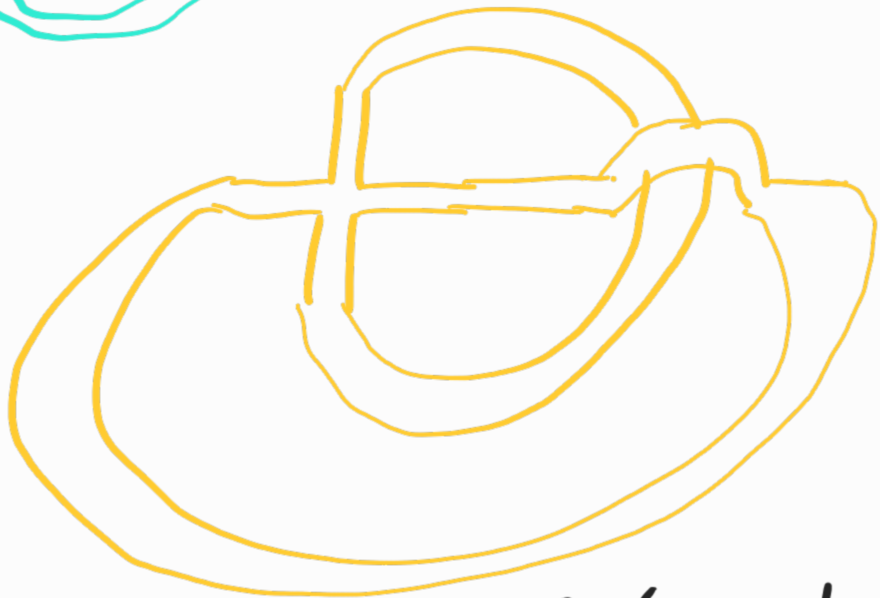


continuity of double arrow

$$\langle M_{ij} M_{jk} M_{kl} M_{li} \rangle_{\frac{1}{N}}$$

$=$

$$\frac{g^2 N^3}{N} = g^2 N^2 = \lambda^2$$



$$g^2 \frac{N}{N} = g^2 = \lambda^2 \frac{1}{N^2}$$

$$\therefore \left\langle \frac{1}{N} \mathrm{Tr}\, M^4 \right\rangle = \lambda^2 \left(2 + \frac{1}{N^2}\right)$$

Euler character

$V = 1, \; E = 2, \; F = 3$

$V - E + F = 2 - 2g = 2$

$\Rightarrow g = 0$

$V = 1, \; E = 2, \; F = 1$

$V - E + F = 2 - 2g = 0$

$\Rightarrow g = 1$

$g=0$ planar diagram
(can be drawn on S^2
which is a compactified
plane $\mathbb{R}^2$)

$g>1$ non-planar diagram

$$\langle \mathcal{O} \rangle = f_0(\lambda) + \frac{1}{N^2} f_1(\lambda) + \frac{1}{N^4} f_2(\lambda) + \cdots$$

$$= \sum_{g=0}^{\infty} f_g(\lambda) \frac{1}{N^{2g}}$$

Here we have generalized our
matrix model to

$$S = N\left(\frac{M^2}{2} + a\frac{M^4}{4} + b\frac{M^6}{6} + \cdots\right)$$

$$\mathcal{O} = \frac{1}{N} \operatorname{Tr} M^p$$

$$\langle \mathcal{O} \rangle \equiv \frac{1}{Z} \int dM \, \mathcal{O} e^{-S}$$

"Triangulation" (simplicial decomposition)
of Riemann surfaces
local coordination number # $\rightarrow$ local metric

Dynamically triangulated random surfaces (DTRS)

Easily visualizable example of DTRS

$$S = N\,\mathrm{Tr}\left(\frac{M^2}{2} + \alpha M^3\right)$$

The back side should also be coloured. (see model)

$\Rightarrow F = 4, \quad V = 4, \quad E = 6$

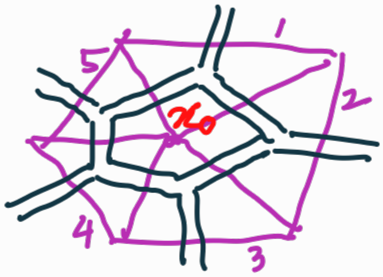
$V - E + F = 2 \Rightarrow S^2$ (genus $= 0$)

Dual picture

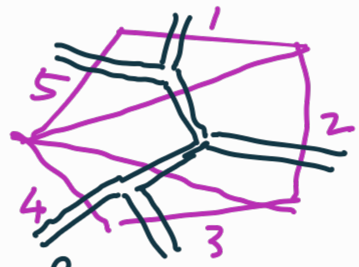
$V' = 4, \; F' = 4, \; E' = 6$

$V' - E' + F' = F - E + V = V - E + F = 2$

More interesting example



Back



M^3 Feynman diagram $\xRightarrow{\text{dual}}$ Triangulation of Riemann surface

Metric $g(x)$

$$\boxed{\sqrt{g(x_0)}\,R(x_0) = \frac{2\pi}{p}\left(4 - (p-2)(q-2)\right)\delta(x - x_0)} \quad (1)$$

where q p-gons meet at x_0

For triangulations $p = 3 \Rightarrow$ RHS $= 2\pi/3\,(6 - q)$

In the figure $q = 5$ $\therefore \sqrt{g(x_0)}\,R(x_0) = 2\pi/3$ (note $+$ve)

Home Work 1: Sum over contribution from all vertices and show $\int \sqrt{g(x_0)}\,R(x_0) = 8\pi$

Note that for the tetrahedron, at each vertex

$q = 3, \; p = 3 \Rightarrow \sqrt{g}\,R = \frac{2\pi}{3}(4 - 1.1) = 2\pi\,\delta(x - x_i)$, $i = 1, 2, 3, 4$

There are 4 vertices hence

$$\int \sqrt{g}\,R\,d^2x = 4 \times 2\pi = 8\pi \quad \left(\text{recall } \frac{1}{4\pi}\int \sqrt{g}\,R = \chi = 2 - 2g\right)$$

(c) Yang Mills Theory

$\int \mathcal{D}A_\mu^{ij}(x)\, e^{-S}$ $\quad S = \frac{1}{g_{YM}^2}\int d^4x\, \mathrm{Tr}\, F_{\mu\nu}^2$ $\quad F_{\mu\nu}^{ij} = \partial_\mu A_\nu^{ij} - \partial_\nu A_\mu^{ij} - i\,[A_\mu, A_\nu]^{ij}$

Clearly this is also a matrix model (multi-matrix model)

Yang-Mills theory is complicated. Furthermore, asymptotic freedom $\Rightarrow$ dimensional transmutation; the coupling constant merely serves to define an energy scale $\Lambda_{QCD} = \Lambda e^{-\frac{1}{g_{YM}^2(\Lambda)}}$

In particular $G^{(n)}(p_i, g_{YM}, \Lambda) = \Lambda^{\#}\, \bar{G}^{(n)}(\bar{p}_i)$ = indep of g_{YM}

$\bar{p}_i = p_i/\Lambda_{QCD}$

$G^{(n)}(\bar{p}_i)$ are universal (no dependence on any small parameter)

'tHooft proposed $\frac{1}{N}$ as a small parameter to compute such correlators.

There are many examples where this has succeeded (even though $N = 3$ for QCD)

i) Qualitative results in QCD in 4D (Witten)

ii) Quantitative meson spectrum in 2D QCD ('tHooft)

iii) Confinement – deconfinement phase transition in SYM in 4D (Shiraz et al)

iv) 2D lattice gauge theory (Gross-Witten-Wadia)