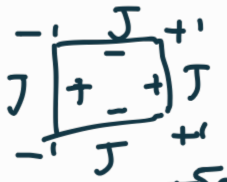
Ising model + gravity

Usual Ising



$e^{-\beta\mathcal{H}} = e^{\beta J \sum_{\langle ij\rangle} S_i S_j}$

$= e^{\beta J\cdot 4}$ + + + +

$+ e^{\beta\cdot 0}$ − + +

\+ − + −

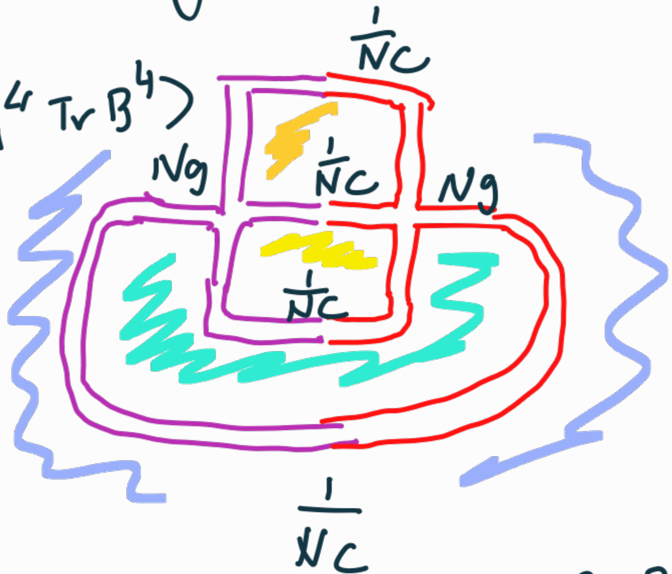
− − + +

correspondence to A, B model

$\mathcal{H} = -J\sum_{\langle ij\rangle} S_i S_j$

$$N\,\mathrm{Tr}\left(A^2+B^2-g\left(A^4+B^4\right)+cAB\right)$$

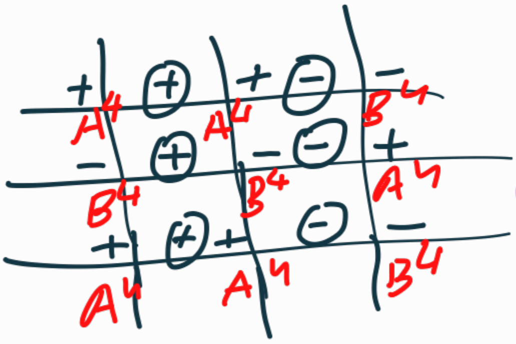
$\langle \mathrm{Tr}\,A^4\,\mathrm{Tr}\,B^4\rangle$



$$\left(\frac{1}{Nc}\right)^4 N^2 g^2 N^4 = \frac{g^2}{c^4}N^2$$

$$= e^{-\beta J}N^2$$

$$\langle(\mathrm{Tr}\,A^4)^2\rangle \Rightarrow g^2N^2 = e^{\beta J}N^2$$



Kazakov 1986

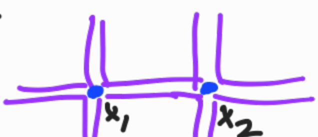
Staudacher

$$Z_n(\beta) = \sum_{\{G^{(n)}\}}\sum_{\{\sigma\}} e^{\beta G^{(n)}_{ij}\sigma_i\sigma_j}$$

$\sigma_i = \pm 1$, $i = 1, \ldots n$ are Ising spins

Sum over Feynman diagrams $\longleftrightarrow \int \mathcal{D}g_{\alpha\beta}(\sigma)$

$M_{ij}(x) \Rightarrow \int_x \mathrm{Tr}(\partial_x M)^2$



$\int dx_1 dx_2\, G(x_1,x_2) \Rightarrow \int \mathcal{D}X(\sigma)\, e^{S[X]}$

$e^{-S} \Rightarrow \int_{x_1}\mathrm{Tr}\,M^4(x_1)\int_{x_2}\mathrm{Tr}\,M^4(x_2)$

$S[X] = \int_{\sigma_1\sigma_2} G(X(\sigma_1), X(\sigma_2))$

$\sim \int \partial_\alpha X\,\partial_\beta X\, g^{\alpha\beta}(\sigma)\, d^2\sigma$

$\therefore \int \mathcal{D}M_{ij}(x)\, e^{-S[M]}$ $\quad S[M] = \int dx \left(\mathrm{Tr}\,(\partial_x M)^2 - \mathrm{Tr}\, V(M(x)) \right)$

$\Longleftrightarrow \quad \int \mathcal{D}g_{\alpha\beta}(\sigma)\, \mathcal{D}X(\sigma)\, e^{-S[X,g]}$

$$S[X,g] = \int \partial_\alpha X \partial_\beta X\, g^{\alpha\beta}(\sigma) \sqrt{g}\, d^2\sigma$$

2D gravity + matter

note the absence of $\sqrt{g} R$ (we have induced gravity here) → we will come back to such terms later in the context of JT gravity (we will have $\sqrt{g} R \phi$ etc.)

$\Rightarrow$ Matrix models $\longleftrightarrow$ 2D quantum gravity (+ matter)

Relation to string theory

Nambu-Goto $\Rightarrow$

$\sum_{\text{surfaces}} e^{-S}$ $\quad S = \text{area} = \int d^2\sigma \sqrt{\hat{g}}$

$\hat{g}_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\mu$

Polyakov

$\mathcal{M}$

$\int_{\mathcal{M}} \mathcal{D}g_{\alpha\beta}(\sigma)\, \mathcal{D}X^\mu(\sigma)\, e^{-S[X,g]}$

$$S[X,g] = \int d^2\sigma \sqrt{g}\, g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\mu$$

Quantitative calculations in 1-matrix model at large N

Consider $Z = \underbrace{\int \prod_{i\leq j} dM^R_{ij} \prod_{i<j} dM^I_{ij}}_{[dM]} e^{-S}$ $\quad S = N\,\mathrm{Tr}\left(\frac{M^2}{2} + gM^4\right) = N\,\mathrm{Tr}\,V(M)$

Eigenvalues

$$Z = \int \prod_{i=1}^{N} d\lambda_i\, \Delta^2(\lambda) \exp\left[-N\sum_{i=1}^{N} V(\lambda_i)\right] \qquad \Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)$$

= van der Monde

$$= \int \prod_{i=1}^{N} d\lambda_i \exp\left[-S_{eff}(\{\lambda_i\})\right]$$

$$S_{eff}(\{\lambda_i\}) = N\sum_{i=1}^{N} V(\lambda_i) - \sum_{i\neq j} \ln|\lambda_i - \lambda_j|$$

$$\longrightarrow N^2\left(\int_{-\infty}^{+\infty} \rho(\lambda) V(\lambda)\, d\lambda - \iint_{\lambda\neq\lambda'} \rho(\lambda)\rho(\lambda') \ln|\lambda - \lambda'|\right) *$$

$$\rho(\lambda) = \frac{1}{N}\sum_{i=1}^{N} \delta(\lambda - \lambda_i)$$

$$\prod_{i=1}^{N} d\lambda_i \longrightarrow \mathcal{D}\rho(\lambda)$$

$$\therefore\ Z = \int \mathcal{D}\rho(\lambda) \exp\left[-N^2 S_0[\rho(\lambda)]\right]$$

$$S_0 = \int d\lambda\, \rho(\lambda) V(\lambda) - \iint_{\lambda\neq\lambda'} d\lambda\, d\lambda' \rho(\lambda)\rho(\lambda') \ln|\lambda - \lambda'|$$

$$\frac{\delta S_0}{\delta\rho} \Rightarrow \quad V(\lambda) = 2\int d\lambda' \rho(\lambda') \ln|\lambda - \lambda'|$$

$$\Rightarrow \frac{1}{2}V'(\lambda) = \fint d\lambda' \frac{\rho(\lambda')}{\lambda - \lambda'} = P\int \frac{\rho(\lambda')d\lambda'}{\lambda - \lambda'}$$

For $V(\lambda) = \frac{\lambda^2}{2} + \alpha\lambda^4$ $\qquad \frac{\lambda}{2} + 2\alpha\lambda^3 = \fint d\lambda' \frac{\rho(\lambda')}{\lambda - \lambda'}$

Solution $\rho(\lambda) = \frac{1}{\pi}\left(\frac{1}{2} + \alpha a_+^2 + 2\alpha\lambda^2\right)\sqrt{a_+^2 - \lambda^2}\,\Theta(a_+^2 - \lambda^2)$

where a_+ satisfies $3\alpha a^4 + a^2 - 4 = 0$

$$a_\pm^2 = -\frac{1}{6\alpha} \pm \frac{1}{6\alpha}\sqrt{1 + 48\alpha}$$

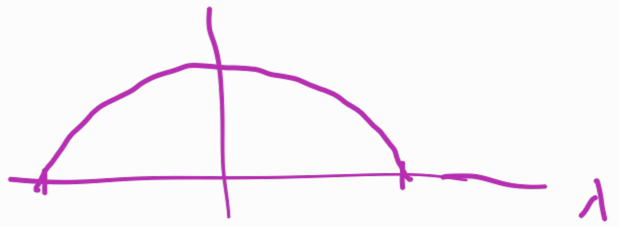
Computation of moments

$$\left\langle \frac{\mathrm{Tr}\,M^p}{N}\right\rangle = \frac{\int [dM]\, \frac{\mathrm{Tr}\,M^p}{N}\, e^{-S(M)}}{\int [dM]\, e^{-S(M)}}$$

$$= \frac{\int[\mathcal{D}\rho(\lambda)]\int d\lambda\,\rho(\lambda)\,\lambda^p\, e^{-S[\rho]}}{\int[\mathcal{D}\rho(\lambda)]\, e^{-S[\rho]}}$$

$$\approx \int d\lambda\,\rho_s(\lambda)\,\lambda^p$$

$\underline{\alpha=0}$ $\qquad \rho_s(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2}\;\theta(4-\lambda^2)$



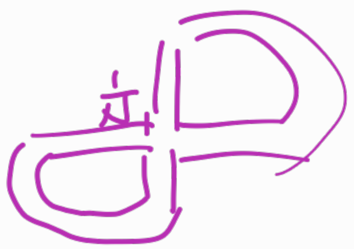
$$\underbrace{(2\pi\rho_s)^2+\lambda^2=4}_{\text{semicircle law}}$$

Compute $\left\langle \frac{\mathrm{Tr}\,M^4}{N}\right\rangle = \frac{1}{2\pi}\int_{-2}^{2} d\lambda\, \lambda^4 \underbrace{\sqrt{4-\lambda^2}}_{2\cos\theta}$ $\qquad \lambda = 2\sin\theta$

$d\lambda \to 2\cos\theta\, d\theta \quad 2^4\sin^4\theta$

$= 2$

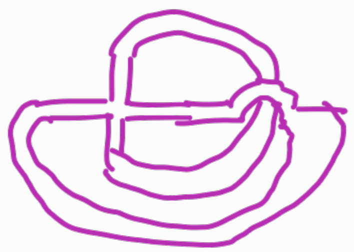
Recall



$\frac{1}{N}\times\frac{1}{N^2}\times N^3 = 1 \quad \times 2$

$\Longrightarrow 2 \checkmark$

$\underline{\text{Non-planar}}$



$\frac{1}{N}\times\frac{1}{N^2}\times N = \frac{1}{N^2}$

$\Rightarrow \left\langle \frac{1}{N}\mathrm{Tr}\,M^4\right\rangle = 2+\frac{1}{N^2}$

Can the $\rho(\lambda)$ theory give the $\frac{1}{N^2}$ term?

No, since $S[\rho] = N^2\left(\int d\lambda\, \rho(\lambda)\, V(\lambda) - \iint d\lambda d\lambda' \rho(\lambda) \rho(\lambda') \ln|\lambda-\lambda'|\right)$

Quadratic in ρ

$$S[\rho_s + \frac{\delta\rho}{N}] = -N^2 \iint d\lambda\, d\lambda'\, \delta\rho(\lambda)\, \delta\rho(\lambda') \ln|\lambda-\lambda'|$$

$\Rightarrow \langle \delta\rho \rangle = 0 \quad \therefore \langle \rho(\lambda) \rangle = \rho_s(\lambda) \Rightarrow 2 \cancel{+ \frac{1}{N^2}}$

$\therefore$ $\rho(\lambda)$ description does not work beyond leading order.

In order to go to subleading order in the (near)-critical theory, there are other methods (e.g. orthogonal polynomials)