Quantitative calculations in 1-matrix model at large N

Consider $Z = \underbrace{\int \prod_{i\le j} dM^R_{ij} \prod_{i<j} dM^I_{ij}}_{[dM]} \, e^{-S}$ $\qquad S = N \,\mathrm{Tr}\left(\frac{M^2}{2} + g M^4\right) = N\,\mathrm{Tr}\, V(M)$

Eigenvalues

$$Z = \int \prod_{i=1}^{N} d\lambda_i \, \Delta^2(\lambda) \exp\left[-N \sum_{i=1}^{N} V(\lambda_i)\right] \qquad \Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j) = \text{van der Monde}$$

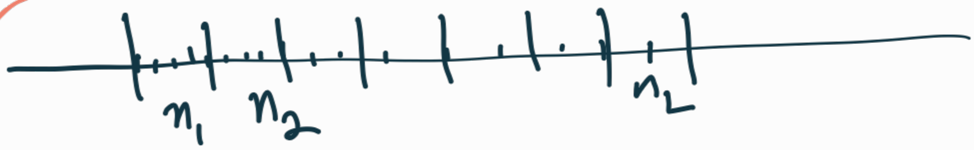
$$= \int \prod_{i=1}^{N} d\lambda_i \exp\left[-S_{eff}(\{\lambda_i\})\right]$$

$$S_{eff}(\{\lambda_i\}) = N \sum_{i=1}^{N} V(\lambda_i) - \sum_{i\neq j}^{N} \ln|\lambda_i - \lambda_j|$$

$$\longrightarrow N^2\left(\int_{-\infty}^{+\infty} \rho(\lambda) V(\lambda)\, d\lambda - \iint_{\lambda\neq\lambda'} \rho(\lambda)\rho(\lambda') \ln|\lambda - \lambda'|\right) *$$

$$\rho(\lambda) = \frac{1}{N}\sum_{i=1}^{N} \delta(\lambda - \lambda_i)$$

$$\prod_{i=1}^{N} d\lambda_i \longrightarrow \mathcal{D}\rho(\lambda)\, J[\rho]$$



$$J = \frac{N!}{n_1!\, n_2! \cdots n_L!}$$

ways to distribute $n_1, n_2, \ldots n_L$ eigenvalues on the ~~real~~ line (split into L intervals) $L \to \infty$ suitably

$$n_r = \tilde\rho(\lambda_r)\Delta(\lambda)$$

$$\ln J = N \ln N - \sum_{r=1}^{L} n_r \ln n_r$$

$$= N\ln N - \sum_{r=1} \tilde\rho(\lambda_r)\Delta\lambda \underbrace{\ln\left(\tilde\rho(\lambda_r)\Delta\lambda\right)}_{\ln\tilde\rho(\lambda_r) + \ln\Delta\lambda}$$

$$= N\ln N - \underbrace{\Delta\lambda \sum_{r=1}^{L} \tilde\rho(\lambda_r)\ln\tilde\rho(\lambda_r)}_{\int d\lambda\, \tilde\rho(\lambda)\ln\tilde\rho(\lambda)} - \underbrace{\Delta\lambda \ln\Delta\lambda \sum_{r=1}^{L}\tilde\rho(\lambda_r)}_{N\ln\Delta\lambda}$$

$$\int \tilde\rho(\lambda)\, d\lambda = \sum n_r = N$$

$$\tilde\rho(\lambda) = N\rho(\lambda)$$

$$J = \exp\left[-N\int d\lambda\, \rho(\lambda)\ln\rho(\lambda)\right]$$

$$* \sum_{i\neq j} \ln|\lambda_i - \lambda_j| = \sum_{i,j} \ln|\lambda_i - \lambda_j| - \sum_i \ln|\lambda_i - \lambda_i|$$

$\ln\varepsilon_\lambda$

$$= N^2 \int d\lambda\, d\mu \ln|\lambda-\mu|\rho(\lambda)\rho(\mu) - N\int d\lambda \ln|\lambda-\lambda|\rho(\lambda)$$

ε_λ: shortest dist. to hold a second eigenval

$$= N^2 \int d\lambda\, d\mu \ln|\lambda-\mu|\,\rho(\lambda)\rho(\mu) - N\int d\lambda\, \rho(\lambda)\ln\frac{1}{N\rho(\lambda)}$$

$$1 = \int_{\varepsilon_\lambda} d\lambda\, \tilde\rho(\lambda) \Rightarrow \rho(\lambda)\varepsilon_\lambda = \frac{1}{N}$$

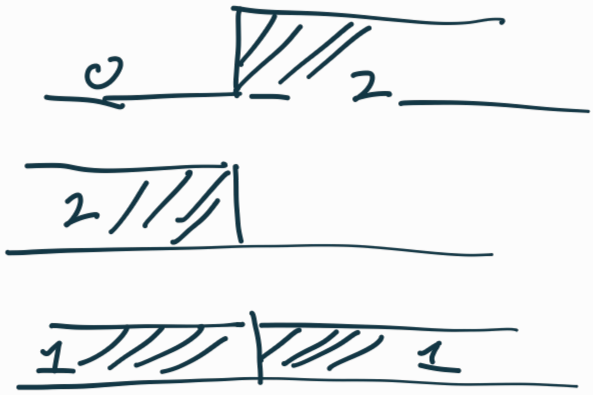
$$= N^2\int d\lambda\, d\mu \ln|\lambda-\mu|\,\rho(\lambda)\rho(\mu) + N\int d\lambda\, \rho(\lambda)\ln\rho(\lambda)$$

where $\int d\lambda\, \tilde\rho(\lambda) = N$ $\quad \tilde\rho(\lambda) = N\rho(\lambda)$

The origin of the combinatorial formula $\dfrac{N!}{\prod_i n_i!}$

Approximate $\rho(\lambda)$ by a Histogram

N=2



$\lambda_1 > 0$
$\lambda_2 > 0$

$\lambda_1 < 0$
$\lambda_2 < 0$

$\left\{ \begin{array}{l} \lambda_1 > 0 \\ \lambda_2 < 0 \end{array} \right.$ or $\begin{array}{l} \lambda_1 < 0 \\ \lambda_2 > 0 \end{array}$

Factor of 2

(More explicitly

$$\int d\lambda_1 \int d\lambda_2 = \left(\int_> d\lambda_1 + \int_< d\lambda_1 \right)\left(\int_> d\lambda_2 + \int_< d\lambda_2 \right)$$

$$= \int_> d\lambda_1 \int_> d\lambda_2 + \int_< d\lambda_1 \int_< d\lambda_2 + \underbrace{\int_> d\lambda_1 \int_< d\lambda_2 + \int_< d\lambda_1 \int_> d\lambda_2}_{\text{same } \rho(\lambda)}$$

More generally, make L bins ($L \to \infty$)

with $\displaystyle\int_{k\text{-th bin}} \rho(\lambda)\, d\lambda = n_k \qquad k = 1, 2, \dots L$

$$\sum_{k=1}^{L} n_k = N$$

$$\prod_k \sum_{n_k=0}^{\cancel{\infty}\ N} \underbrace{\frac{N!}{n_1! \cdots n_L!}}_{J} \longrightarrow \int dn_k\, e^{\ln J} \longrightarrow \int d\rho(\lambda)\, e^{\ln J}$$

$$n_k = \frac{\rho(\lambda_k)}{\varepsilon} \qquad \int \prod_{k=1}^{L} d\rho(\lambda_k)$$

$$\ln J \sim N \int \rho \ln \rho \, d\lambda$$

Another contribution comes from the ε_λ regularization described earlier (of the $\int \rho\rho \ln|\;|$ term)

$$N^2 \int d\lambda\, d\mu\, \rho(\lambda)\rho(\mu) \ln|\lambda - \mu|$$

$$+ N^2 \int d\lambda\, \rho(\lambda) \ln \rho(\lambda)$$

$$\therefore\; S_{ren} = N^2\Big(\int \rho(\lambda)\,V(\lambda)\,d\lambda - \iint_{\lambda\neq\lambda'} d\lambda\, d\lambda'\,\rho(\lambda)\rho(\lambda')\,\ln|\lambda-\lambda'|\Big)$$

$$- \underbrace{N\int \rho(\lambda)\ln\rho(\lambda)\,d\lambda}_{\text{from } J} + \underbrace{N\int \rho(\lambda)\ln\rho(\lambda)\,d\lambda}_{\text{counterterm}}$$

$$\therefore\; Z = \int \mathcal{D}\rho(\lambda)\,\exp\left[-N^2 S_0[\rho(\lambda)]\right]$$

$$S_0 = \int d\lambda\,\rho(\lambda)\,V(\lambda) - \iint_{\lambda\neq\lambda'} d\lambda\,d\lambda'\,\rho(\lambda)\rho(\lambda')\ln|\lambda-\lambda'|$$

$$\frac{\delta S_0}{\delta\rho} \Rightarrow \quad V(\lambda) = 2\int d\lambda'\,\rho(\lambda')\ln|\lambda-\lambda'|$$

$$\Rightarrow \frac{1}{2}V'(\lambda) = ⨍ d\lambda'\,\frac{\rho(\lambda')}{\lambda-\lambda'} = P\int\frac{\rho(\lambda')d\lambda'}{\lambda-\lambda'}$$

For $V(\lambda) = \frac{\lambda^2}{2} + \alpha\lambda^4$ $\qquad \frac{\lambda}{2} + 2\alpha\lambda^3 = ⨍ d\lambda'\,\frac{\rho(\lambda')}{\lambda-\lambda'}$

Solution $\rho(\lambda) = \frac{1}{\pi}\left(\frac{1}{2} + \alpha a_+^2 + 2\alpha\lambda^2\right)\sqrt{a_+^2-\lambda^2}\,\Theta(a_+^2-\lambda^2)$

where a_+ satisfies $3\alpha a^4 + a^2 - 4 = 0$

$$a_\pm^2 = -\frac{1}{6\alpha} \pm \frac{1}{6\alpha}\sqrt{1+48\alpha}$$

Proof: Construct resolvent

$$R(z) = \frac{1}{N}\mathrm{Tr}(z-M)^{-1} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{z-\lambda_i} = \int d\lambda\,\frac{\rho(\lambda)}{z-\lambda}$$

$z\in\mathbb{C}$

By Sokhotski-Plemelj theorem ($x, \varepsilon \in \mathbb{R}$)

$$R(x \pm i\varepsilon) = P\int d\lambda \, \frac{\rho(\lambda)}{x-\lambda} \mp i\pi\,\rho(x) \qquad (\varepsilon \to 0)$$

[Proof: $R(x+i\varepsilon) = \int d\lambda \, \frac{\rho(\lambda)}{x-\lambda+i\varepsilon} = \int d\lambda \, \frac{\rho(\lambda)(x-\lambda-i\varepsilon)}{(x-\lambda)^2+\varepsilon^2}$

$$= \int d\lambda \, \rho(\lambda) \underbrace{\frac{(x-\lambda)}{(x-\lambda)^2+\varepsilon^2}}_{\chi_{\mathbb{R}-\{x\}}} - i\int d\lambda \, \rho(\lambda) \underbrace{\frac{\varepsilon}{(x-\lambda)^2+\varepsilon^2}}_{\pi\delta(x-\lambda)}$$

~~Note~~ $\frac{x-\lambda}{(x-\lambda)^2+\varepsilon} \to 1$ as $|(x-\lambda)| \gg \varepsilon$

$\to 0$ as $|x-\lambda| \ll \varepsilon$ QED]

$$\therefore \; R(x \pm i\varepsilon) = \tfrac{1}{2}V'(x) \mp i\pi\rho(x) \quad \text{——} (*)$$

Ansatz $R(z) = \frac{1}{2}V'(z) - P(z)\sqrt{z^2-a^2}$ $\quad P(z) = p_0 + p_1 z + p_2 z^2 + \cdots$

[Sq. root branch cut is suggested by the 2-sheeted structure of (*). The end pts of the branch cut are chosen symmetrically ($\pm a$) since $V(\lambda)$ is even. $a<\infty$ is put in by hand, it will turn out so by normalization of $\rho(\lambda)$]

Note that $R(z) = \int_{-a}^{a} d\lambda \, \frac{\rho(\lambda)}{z-\lambda} \xrightarrow{|z|\to\infty} \frac{1}{z}\int_{-a}^{+a} d\lambda\,\rho(\lambda) = \frac{1}{z}$

From ansatz, as $|z| \to \infty$

$$R(z) = \frac{z}{2} + 2\alpha z^3 - (p_0 + p_1 z + p_2 z^2)\, z\left(1 - \frac{a^2}{2z^2} - \frac{1}{8}\frac{a^4}{z^4} + \cdots\right)$$

$$= \frac{z}{2} + 2\alpha z^3 - \Big\{ p_0\Big(z - \frac{a^2}{2z}\Big) + p_1\Big(z^2 - \frac{a^2}{2} - \frac{1}{8}\frac{a^4}{z^2} + \cdots\Big) + p_2\Big(z^3 - \frac{a^2}{2}z - \frac{a^4}{8}\frac{1}{z} + \cdots\Big)\Big\} = \frac{1}{z}$$

$\Rightarrow (z^3) \Rightarrow 2\alpha - p_2 = 0 \quad p_2 = 2\alpha$

$(z^2) \Rightarrow p_1 = 0 \qquad (z) \Rightarrow \frac{1}{2} - p_0 + \frac{p_2 a^2}{2} = 0 \qquad (\frac{1}{z}) \Rightarrow \frac{p_0 a^2}{2} + \frac{p_2 a^4}{8} = 1$

$\frac{1}{2} - p_0 + \alpha a^2 = 0$

$p_0 = (\alpha a^2 + \frac{1}{2})$

$$\frac{a^2}{2}\left(\alpha a^2 + \tfrac{1}{2}\right) + \frac{\alpha}{4}a^4 = 1$$

$$\Rightarrow \frac{3\alpha a^4}{4} + \frac{a^2}{4} - 1 = 0 \Rightarrow 3\alpha a^4 + a^2 - 4 = 0$$

$$\Rightarrow a_\pm^2 = \frac{1}{6\alpha}\left(-1 \pm \sqrt{1+48\alpha}\right) \xrightarrow{+\text{ root}} a^2 = \frac{1}{6\alpha}\left(\sqrt{1+48\alpha} - 1\right)$$

$$\therefore R(z) = z/2 + 2\alpha z^3 - \left(\left(\alpha a^2 + \tfrac{1}{2}\right) + 2\alpha z^2\right)\sqrt{z^2-a^2}$$

$$\underbrace{R(x+i\varepsilon) - R(x-i\varepsilon)}_{-2i\pi\rho(x)} = 0 \quad \text{if } x^2 > a^2$$

$$= -\left(\tfrac{1}{2} + \alpha a^2 + 2\alpha x^2\right)\sqrt{a^2-x^2}\,(2i)$$

$$\Rightarrow \rho(x) = \frac{1}{\pi}\left(\tfrac{1}{2} + \alpha a^2 + 2\alpha x^2\right)\sqrt{a^2-x^2}\;\Theta(a^2-x^2)$$

QED

Recall $\mathcal{Z} = \int [\mathcal{D}\rho]\, e^{-N^2 S_{eff}[\rho]}$

$$S_{eff} = \int_{-a}^{a} d\lambda\, \rho(\lambda) V(\lambda) - \underset{\lambda\neq\lambda'}{\int_{-a}^{a} d\lambda \int_{-a}^{a} d\lambda'}\, \rho(\lambda)\rho(\lambda') \ln|\lambda-\lambda'|$$

We wish to evaluate this at $\rho(\lambda) = \rho_s(\lambda)$ $s \to$ saddle

<u>Note</u>: $\displaystyle \int_{-a}^{a} \frac{d\lambda'\, \rho(\lambda')}{\lambda-\lambda'} = \frac{1}{2}V'(\lambda) \xrightarrow{\int_0^\lambda d\lambda} \int_{-a}^{a} d\lambda' \rho(\lambda')\left(\ln|\lambda-\lambda'| - \ln|\lambda'|\right)$

$$= \frac{1}{2}\left(V(\lambda) - V(0)\right) = \frac{1}{2}V(\lambda)$$

$$\Rightarrow \int_{-a}^{a} d\lambda\, \rho(\lambda) \underbrace{\left[\int_{-a}^{a} d\lambda' \rho(\lambda') \ln|\lambda-\lambda'|\right]}_{\frac{1}{2}V(\lambda) + \int_{-a}^{a} d\lambda' \rho(\lambda') \ln|\lambda'|}$$

$$= \frac{1}{2}\int_{-a}^{a} d\lambda\, \rho(\lambda) V(\lambda) + \int_{-a}^{a} d\lambda'\, \rho(\lambda') \ln|\lambda'| \qquad \left(\text{using } \int d\lambda\, \rho(\lambda) = 1\right)$$

$$\therefore S_{eff}[\rho_s(\lambda)] = \frac{1}{2}\int_{-a}^{a} d\lambda\, \rho(\lambda) V(\lambda) - \int_{-a}^{a} d\lambda\, \rho(\lambda) \ln|\lambda|$$

$$= \int_0^a d\lambda\, \rho(\lambda) V(\lambda) - 2\int_0^a d\lambda\, \rho(\lambda) \ln|\lambda|$$

$$\mathcal{Z} \sim e^{-N^2 S_{eff}[\rho_s]} \equiv e^{-\mathcal{F}}$$

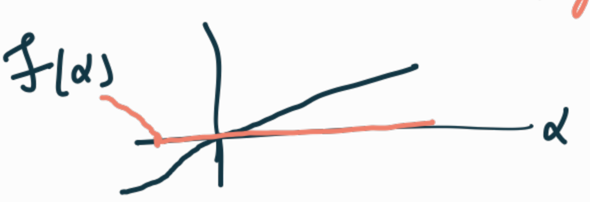
$$\mathcal{F} = N^2 S_{eff}[\rho_s] \Rightarrow \frac{\mathcal{F}}{N^2} = 2\alpha\; {}_3F_2\left(1, 1, \tfrac{3}{2}; 2, 4; -48\alpha\right)$$

$$\lim_{\alpha\to\alpha_c} \partial_\alpha^{(3)} \mathcal{F}_{non\text{-}an}(\alpha) = 4608\sqrt{3}\,(\alpha-\alpha_c)^{-1/2} \qquad \alpha_c = -\frac{1}{48}$$

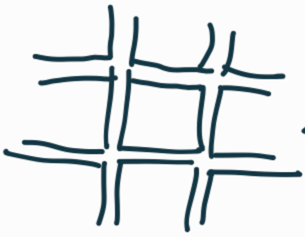
(cf. $\sqrt{1+48\alpha}$)

i.e. $$\partial_\alpha^{(3)}\mathcal{F}(\alpha) = \underbrace{P(\alpha)}_{\text{terms regular at } \alpha\to\alpha_c} + 4608\sqrt{3}\,(\alpha-\alpha_c)^{-1/2}$$

$\left(\sim (\alpha-\alpha_c)^n,\ (\alpha-\alpha_c)^{n+\frac{1}{2}}\right\}\ n=0,1,\ldots)$



Partition fn for $S = N\,\mathrm{Tr}\left(\frac{M^2}{2} + aM^4\right)$



$$\sum_{V,L} (Na)^V \left(\frac{1}{N}\right)^P N^L n_{V,L}$$

$$= \sum_{g=0}^{\infty} N^{2-2g} \underbrace{\sum_V a^V n_{V,g}}_{f_g(a)}$$

$L = F$
$P = E$
$V = V$
$V - E + F = \chi$
$= 2 - 2g$

$V = 2$
$P = 2V = 4$
$L = 4$ or 2

$f_g(a)$ has a critical point near which

$$f_g(a) \sim (a - a_c)^{5\chi_g/4}$$

$g = 0 \quad f_0(a) \sim (a - a_c)^{5/2} \quad f_0''' \to \infty$

Then,

$$\langle\langle V \rangle\rangle_g \equiv \sum_V V\, a^V n_{V,g}$$

$$= a\frac{\partial}{\partial a} f_g(a)$$

$$\sum_V a^V n_{V,g} \quad \downarrow a\frac{\partial}{\partial a} \quad \sum_V V a^V n_{V,g}$$

regular at $a = a_c$

$$f_0(a) = P_0(a) + \alpha(a-a_c)^{5/2}$$

$$a f_0'(a) = a\left(P_0'(a) + \frac{5\alpha}{2}(a-a_c)^{3/2}\right)$$

$$\langle\langle V^2 \rangle\rangle_0 = \left(a\frac{\partial}{\partial a}\right)^2 f_0(a) = a\left(P_0'(a) + \frac{5\alpha}{2}(a-a_c)^{3/2}\right) + a\left(P_0''(a) + \frac{15}{4}\alpha(a-a_c)^{\frac{1}{2}}\right)$$

$$\langle\langle V^3 \rangle\rangle = \left(a\frac{\partial}{\partial a}\right)^3 f_0(a) = a\left[a^2\left\{P_0'''(a) + \frac{15\alpha}{8}(a-a_c)^{-1/2}\right\} + \cdots\right]$$

$$\langle V^3 \rangle = \frac{\langle\langle V^3 \rangle\rangle}{f_0(a)} = \frac{a^3\left[P_0'''(a) + \frac{15\alpha}{8}(a-a_c)^{-1/2} + \cdots\right]}{P_0(a) + \alpha(a-a_c)^{5/2}}$$

Here $\cdots$ represents terms that remain finite as $a \to a_c$

$\therefore \langle V^3 \rangle \longrightarrow \infty$ as $a \to a_c$

$\Rightarrow$ # vertices effectively diverges as $a \to a_c$

Note 3rd order phase transition

In the dual lattice $V \longrightarrow F$

$\Rightarrow \langle F^3 \rangle \to \infty$

$\Rightarrow$ number of "triangles" $\to \infty$

This is a continuum limit of triangulated surfaces.

Computation of moments

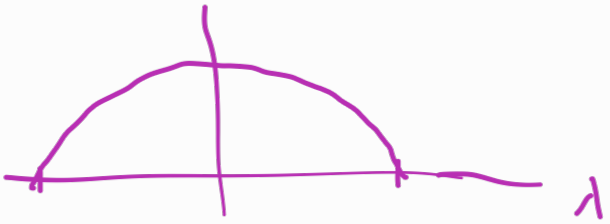
$$\left\langle \frac{\mathrm{Tr}\, M^p}{N} \right\rangle = \frac{\int [dM]\, \frac{\mathrm{Tr}\, M^p}{N}\, e^{-S(M)}}{\int [dM]\, e^{-S(M)}}$$

$$= \frac{\int [\mathcal{D}\rho(\lambda)] \int d\lambda\, \rho(\lambda) \lambda^p\, e^{-S[\rho]}}{\int [\mathcal{D}\rho(\lambda)]\, e^{-S[\rho]}}$$

$$\approx \int d\lambda\, \rho_s(\lambda)\, \lambda^p$$

$\underline{\alpha = 0}$

$$\rho_s(\lambda) = \frac{1}{2\pi} \sqrt{4-\lambda^2}\; \theta(4-\lambda^2)$$



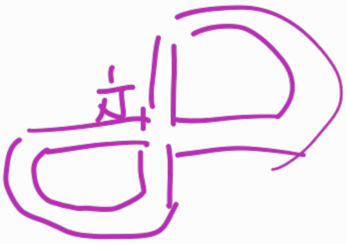
$$\underbrace{(2\pi \rho_s)^2 + \lambda^2 = 4}_{\text{semi circle law}}$$

Compute
$$\left\langle \frac{\mathrm{Tr}\, M^4}{N} \right\rangle = \frac{1}{2\pi} \int_{-2}^{2} d\lambda\, \lambda^4 \underbrace{\sqrt{4-\lambda^2}}_{2\cos\theta} \qquad \lambda = 2\sin\theta$$

$$d\lambda \to 2\cos\theta\, d\theta \qquad \lambda^4 \to 2^4 \sin^4\theta$$

$$= 2$$

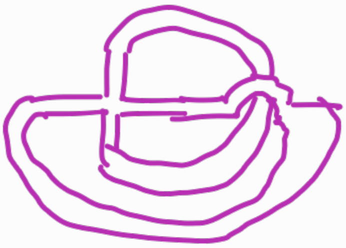
Recall



$$\frac{1}{N} \times \frac{1}{N^2} \times N^3 = 1 \quad \times 2$$

$$\Longrightarrow 2 \;\checkmark$$

$\underline{\text{Non-planar}}$



$$\frac{1}{N} \times \frac{1}{N^2} \times N = \frac{1}{N^2}$$

$$\Rightarrow \left\langle \frac{1}{N} \mathrm{Tr}\, M^4 \right\rangle = 2 + \frac{1}{N^2}$$

Can the $\rho(\lambda)$ theory give the $\frac{1}{N^2}$ term?

No, since $S[\rho] = N^2\left(\int d\lambda\, \rho(\lambda)\, V(\lambda) - \iint d\lambda d\lambda' \rho(\lambda) \rho(\lambda') \ln|\lambda-\lambda'|\right)$

Quadratic in ρ

$$S[\rho_s + \frac{\delta\rho}{N}] = -N^2 \iint d\lambda\, d\lambda'\, \delta\rho(\lambda)\, \delta\rho(\lambda') \ln|\lambda-\lambda'|$$

$\Rightarrow \langle \delta\rho \rangle = 0 \quad \therefore \langle \rho(\lambda) \rangle = \rho_s(\lambda) \Rightarrow 2 \cancel{+ \frac{1}{N^2}}$

$\therefore$ $\rho(\lambda)$ description does not work beyond leading order.

In order to go to subleading order in the (near)-critical theory, there are other methods (e.g. orthogonal polynomials) or loop equations.