Non-planar diagrams : orthogonal polynomials

Consider the Gaussian model (GUE)

Note the following identities involving Hermite fns

$$H_n(x) = (-1)^n e^{x^2} \partial_x^n e^{-x^2}$$

$$= \{1, 2x, 4x^2-2, 8x^3-12x, \dots\}$$

which satisfy orthogonality relations

$$\int_{-\infty}^{+\infty} dx\, e^{-x^2} H_n(x) H_m(x) = 2^n \sqrt{\pi}\, n!\, \delta_{mn}$$

Monic form $P_n(x) = \{1, x, x^2 - \frac{1}{2}, x^3 - \frac{3}{2}x, \dots\}$

(Highest power has coefficient 1)

Lemma :

VanderMonde $\Delta_N^2 = \begin{pmatrix} P_0(\lambda_1) & P_1(\lambda_1) & \cdots & P_{N-1}(\lambda_1) \\ \vdots & & & \\ P_0(\lambda_N) & P_1(\lambda_N) & \cdots & P_{N-1}(\lambda_N) \end{pmatrix}^2$

Hint : $N = 3$

$$\det \begin{pmatrix} 1 & x_1 & x_1^2 - \frac{1}{2} \\ 1 & x_2 & x_2^2 - \frac{1}{2} \\ 1 & x_3 & x_3^2 - \frac{1}{2} \end{pmatrix} = \det \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix}$$

$$= a\,(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$$

($\because$ the determinant clearly vanishes if $x_i = x_j$)

What is a ? $a = -1$ (e.g. compare the coeff. of $x_1^2 x_2$)

$$\therefore Z_N = \int \prod dx_i\, e^{-\sum x_i^2} \Delta^2(x) = \sum_{\sigma, \tau \in S_N} \mathrm{sgn}(\sigma)\, \mathrm{sgn}(\tau) \prod_i \int dx_i\, e^{-\sum x_i^2} P_{\sigma_i - 1}(x_i)\, P_{\tau_i - 1}(x_i)$$

$$\boxed{Z_N = \sum_{\sigma \in S_N} \prod_{i=1}^{N} h_{\sigma_i} = N!\, h_0 h_1 \cdots h_{N-1}} \quad (1)$$

where we used the notation

$$\int dx\, e^{-x^2} P_n(x) P_m(x) = h_n \delta_{nm}$$

Let's generalize

$$e^{-x^2} \longrightarrow e^{-NV(x)}$$

Suppose we have a system of orthogonal polynomials which satisfy

$$\int dx\, e^{-NV(x)} P_n(x) P_m(x) = h_n \delta_{nm}$$

(1) continues to hold :

$$Z_N = \int \prod_i dx_i\, e^{-N\sum V(x_i)} \Delta^2(x) = N! \prod_{i=0}^{N-1} h_i$$

Lemma : $\boxed{h_n = \underline{R_n} h_{n-1}}$ $n \geq 0$ (2)

where we used the following 3-term recursion relation satisfied by the orthogonal polynomials

$$x P_n(x) = P_{n+1}(x) + \underline{R_n} P_{n-1}(x) \; (n \geq 1), \quad x P_0(x) = S_0 P_1(x)$$

(we have assumed that $V(x)$ = even, hence $P_0, P_2, \cdots$ = even
$P_1, P_3, \cdots$ = odd)

Proof of (2):

$$\int d\mu(\lambda)\, \lambda\, P_{n-1}(\lambda)\, P_n(\lambda) = \langle P_n + \# P_{n-2}, P_n \rangle = h_n \qquad d\mu(\lambda) \equiv e^{-NV} d\lambda$$

But LHS is also $= \int d\mu(\lambda)\, P_{n-1}(\lambda)\, (R_n P_{n-1} + P_{n+1})$

$$= R_n h_{n-1} \qquad \underline{\text{QED}}$$

$$\boxed{\therefore Z_N = N!\, h_0^N\, R_1^{N-1} R_2^{N-2} \cdots R_{N-1}} \quad \text{— (3)}$$

We will also use

$$\boxed{n h_{n-1} = \int d\mu(\lambda)\, P_n'(\lambda)\, P_{n-1} = N \int d\mu(\lambda)\, V'(\lambda)\, P_n(\lambda)\, P_{n-1}(\lambda)} \quad \text{— (4)}$$

(The first step follows from

$$P_n'(\lambda) = n P_{n-1}(\lambda) + O(\lambda^{n-3}) = n P_{n-1}(\lambda) + \sum_{l \le n-3} \# P_l(\lambda)$$

The 2nd step follows by partial integration

$$-\int_\lambda \left(-N V'(\lambda)\, P_{n-1}(\lambda) + \underbrace{P_{n-1}'(\lambda) \Big) P_n(\lambda)}_{=0}\, e^{-NV(\lambda)}\right)$$

Take $V(\lambda) = \frac{\lambda^2}{2} + \alpha \lambda^4$

Then $V'(\lambda) = \lambda + 4\alpha \lambda^3$

$$\text{RHS of (4)} = N \int d\lambda\, e^{-NV} (\lambda + 4\alpha\lambda^3)\, P_n(\lambda)\, P_{n-1}(\lambda)$$

$$= N \left(h_n + 4\alpha \langle \lambda^3 P_n, P_{n-1} \rangle \right)$$

$$P_n \xrightarrow{\lambda} P_{n+1} + R_n P_{n-1} \xrightarrow{\lambda} (P_{n+2} + R_{n+1} P_n) + R_n (P_n + R_{n-1} P_{n-2})$$

$$\xrightarrow{\lambda} P_{n+3} + R_{n+2} P_{n+1} + (R_{n+1} + R_n)(P_{n+1} + R_n P_{n-1}) + R_n R_{n-1} (P_{n-1} + R_{n-2} P_{n-3})$$

$$\therefore \langle \lambda^3 P_n, P_{n-1} \rangle = \left((R_{n+1} + R_n) R_n + R_n R_{n-1} \right) h_{n-1}$$

$$\therefore \quad n h_{n-1} = N \left(\underbrace{h_n}_{R_n h_{n-1}} + 4\alpha R_n (R_{n-1} + R_n + R_{n+1}) h_{n-1} \right)$$

$$\Rightarrow \quad \frac{n}{N} = R_n + 4\alpha R_n (R_{n-1} + R_n + R_{n+1}) \quad \text{— (5)}$$

In the large N limit $\frac{n}{N} =: \xi$ continuous

$$R_n = r(\xi) \qquad R_{n\pm 1} = r(\xi \pm \varepsilon) \qquad \varepsilon = \frac{1}{N}$$

$\Rightarrow$ (5) gives (for $\varepsilon \to 0$) PLANAR

$$\xi = r(\xi) + 4\alpha\, r(\xi)(3 r(\xi))$$
$$= r + 12\alpha r^2 \equiv W(r)$$

This matches with (4.20) Anninos (for $s=0$)

$$\Rightarrow \quad r(\xi, \alpha) = \frac{-1 + \sqrt{1 + 48\alpha\xi}}{24\alpha} = 2 + \sqrt{1 - \frac{\alpha}{\alpha_c}(1 - \delta\xi)}$$
$$= 2 + \sqrt{\alpha_c - \alpha}\, \delta r$$

where $\delta r \to 0$ as $\delta\xi \to 0$

Next order: $R_n = r(\xi) = r_0(\xi) + \varepsilon^2 r_2(\xi) + \cdots$

$$\xi = r_0(\xi) + \varepsilon^2 r_2(\xi) + 4\alpha (r_0(\xi) + \varepsilon^2 r_2(\xi))$$
$$(3(r_0(\xi) + \varepsilon^2 r_2(\xi)) + \varepsilon^2 r_0''(\xi))$$

$$-\xi + r_0(\xi) + 12\alpha r_0^2 + \varepsilon^2 [r_2(\xi) + 4\alpha \{4 r_0 r_2 + r_0 r_0''\}] = 0$$

$$\therefore \quad \boxed{r_0(\xi, \alpha) = \frac{-1 + \sqrt{1 + 48\alpha\xi}}{24\alpha}} = \frac{-1 + \sqrt{\left(1 - \frac{\alpha}{\alpha_c}\right)\xi}}{24\alpha}$$

$$\Big\downarrow \xi = 1$$

$$\frac{-1 + \# \sqrt{\Delta\alpha}}{24(\alpha_c + \Delta\alpha)}$$

$$\sim +2 + \# \sqrt{\Delta\alpha}$$

$$\Rightarrow \quad r_2(\xi)(1 + 24\alpha r_0(\xi)) = -4\alpha\, r_0(\xi)\, r_0''(\xi)$$

$$r_2(\xi) = \frac{96\alpha^2 r_0(\xi)}{(1 + 48\alpha\xi)^2}$$

Note for later

At $\xi \to 1$ $\quad r_0(\xi, \alpha)$ is non-analytic at $\alpha = \alpha_c$

$$\alpha_c \equiv -\frac{1}{48}$$

Same is true for $r_2(\xi, \alpha)$

$$\ln Z = N\ln N - N + N\ln h_0 + \sum_{n=0}^{N-1}(N-n)\ln R_n$$

$$\ln Z(\alpha) - \ln Z(0) = N\,\ln\left(h_0(\alpha) - \ln h_0(0)\right)$$
$$+\sum(N-n)\left(\ln R_n(\alpha) - \ln R_n(0)\right)$$

Euler-McLaurrein formula:

$$\frac{1}{N}\sum_{n=1}^{N} f\left(\frac{n}{N}\right) = \int_0^1 d\xi\, f(\xi) + \frac{1}{2N}\, f(\xi)\Big|_0^1$$
$$+\sum_{n=1}^{p-1}\frac{B_{2n}}{(2n)!}\,\frac{1}{N^{2n}}\, f(\xi)^{(2n-1)}\Big|_0^1 + R_N$$

$B_2 = \frac{1}{6}$

Define $R_n =: r(\xi)$ $\quad \xi = \frac{n}{N}$

$$\Rightarrow \sum_{n=0}^{N-1}(N-n)\ln R_n(\alpha) = N^2\int_0^1 d\xi\,(1-\xi)\ln r(\xi,\alpha)$$

$$\text{"}\ \Big|_{\alpha=0} = N^2 \quad \text{"}\ \underbrace{\ln r(\xi,\alpha=0)}_{\ln\xi}$$

$\therefore$ $\ln Z(\alpha) - \ln Z(0)$

$$= N^2\left\{\int_0^1 d\xi\,(1-\xi)\ln\left(\frac{r(\xi,\alpha)}{\xi}\right) + \frac{1}{N}\ln\frac{h_0(\alpha)}{h_0(0)}\right.$$
$$\left. + \frac{1}{2N}\left[(1-\xi)\ln\left(\frac{r(\xi,\alpha)}{\xi}\right)\right]_{\xi=0}^{\xi=1} + \frac{1}{12N^2}\left(\left\{(1-\xi)\ln r_0(\xi,\alpha)\right\}' - \left\{\text{"}\right\}_{\alpha=0}\right)_{\xi=0}^{\xi=1}\right\}$$

$$= N^2\left\{\int_0^1 d\xi\,(1-\xi)\ln\left(\frac{r_0(\xi,\alpha) + \varepsilon^2 r_2(\xi,\alpha)}{\xi}\right)\right.$$
$$+\frac{1}{N}\ln\frac{h_0(\alpha)}{h_0(0)} + \frac{1}{2N}\left[(1-\xi)\ln\left(\frac{r_0+\varepsilon^2 r_2}{\xi}\right)\right]_{\xi=0}^{\xi=1}$$
$$\left. + \frac{1}{12N^2}\left\{\left(-\ln\left(r_0+\varepsilon^2 r_2\right) + (1-\xi)\frac{r_0'+\varepsilon^2 r_2'}{r_0+\varepsilon^2 r_2}\right) - (\text{"})\big|_{\alpha=0}\right\}_{\xi=0}^{\xi=1}\right\}$$

(7)

First term is the constant one.

$$+\frac{\mathcal{F}_N(\alpha)-\mathcal{F}_N(0)}{N^2} = -\int_0^1 d\xi\,(1-\xi)\ln\frac{\sqrt{1+48\alpha\xi}-1}{24\alpha\xi}$$

$$\xi\to 0 \quad \ln\frac{24\alpha\xi}{24\alpha\xi}=0$$

No sing. at $\xi\to 0$ $\forall\alpha$

As $\xi\to 1$, say $\xi=1-y$

$$\frac{1}{N^2}\left(\mathcal{F}_N(\alpha)-\mathcal{F}_N(0)\right) \sim -\int_0^1 dy\, y\left(\ln\left(\frac{\sqrt{1+48\alpha(1-y)}-1}{24\alpha(1-y)}\right)\right)$$

The integrand is singular at $\alpha=\alpha_c\equiv-\frac{1}{48}$ as $y\to 0$

$$\Rightarrow\quad y\left(\ln\left(\frac{\sqrt{1-1(1-y)}-1}{-\frac{1}{2}(1-y)}\right)\right)\to y\ln(\sqrt{y}-1)$$

$$\xrightarrow{\partial_y} \ln(\sqrt{y}-1)+\tfrac{1}{2}y\frac{1}{\sqrt{y}}\frac{1}{\sqrt{y}-1} \quad \text{still smooth as } y\to 0$$

$$\xrightarrow{\partial_y} \frac{1}{2}\frac{1}{\sqrt{y}}\frac{1}{\sqrt{y}-1} \quad \underline{\int dy(\ )} \quad \text{still smooth}$$

$$\xrightarrow{\partial_y} y^{-3/2} \xrightarrow{\int dy(\ )} \text{singular}$$

The 3 derivatives w.r.t. y can be traded off in terms of $\left(\frac{\partial}{\partial\alpha}\right)^3$

$$\int_0 dy\, y\frac{\partial^3}{\partial\alpha^3}\left(\ln\left(\sqrt{1-\frac{\alpha}{\alpha_c}(1-y)}-1\right)\right)$$

$$\longrightarrow \int_0 dy\, y\,\partial_\alpha^2\left\{\frac{-\frac{1}{\alpha_c}(1-y)}{2\sqrt{1-\frac{\alpha}{\alpha_c}(1-y)}}\;\frac{1}{\sqrt{1-\frac{\alpha}{\alpha_c}(1-y)}-1}\right\}$$

$$\longrightarrow \int_0 dy\, y\,\partial_\alpha\left\{\frac{-\frac{1}{\alpha_c}(1-y)}{-4\left(1-\frac{\alpha}{\alpha_c}(1-y)\right)^{3/2}}\right\}$$

keeping only potentially singular terms at $y=0$

$$\int_0 dy\,\#\frac{y(1-y)}{\left(1-\frac{\alpha}{\alpha_c}(1-y)\right)^{5/2}} \longleftarrow \underline{\text{singular}}$$

$$\sim \#\int_0 \frac{dy\, y}{(y-(1-\alpha/\alpha_c))^{5/2}} \sim \left(\frac{\alpha}{\alpha_c}-1\right)^{-1/2} \quad \underline{\text{as before}}$$

Actual calculations

$$\mathcal{F}^{(0)}(\alpha) = 2\alpha\ {}_3F_2\left(1,1,\tfrac{3}{2};2,4;-48\alpha\right) \overset{\alpha\to\alpha_c}{\sim} (\alpha-\alpha_c)^{5/2}$$

$$\lim_{\alpha\to\alpha_c} \partial_\alpha^3 \mathcal{F}^{(0)}(\alpha) = 4608\sqrt{3}(\alpha-\alpha_c)^{-1/2}$$

NON-PLANAR (see eq. (7))

Compute $h_0(\alpha)$ $\qquad V(x) = \frac{x^2}{2} + \alpha x^4$

- $$h_0(\alpha) = \int_{-\infty}^{+\infty} dx\, e^{-NV(x)} = \frac{e^{N/32\alpha}}{2\sqrt{2}}\sqrt{\frac{1}{\alpha}}\, K_{\frac{1}{4}}\left(\frac{N}{32\alpha}\right)$$

$$= \sqrt{\frac{2\pi}{N}}\left(1 - \frac{3}{N}\alpha + \frac{105}{2N^2}\alpha^2 + \cdots\right)$$

- $$N^2 \frac{1}{N}\ln\frac{h_0(\alpha)}{h_0(0)} = \frac{N^2}{N}\ln\left(1-\frac{3}{N}\alpha+\cdots\right) = -3\alpha + \cdots \quad \text{— (a)}$$

$$N^2\varepsilon^2\int_0^1 d\xi\,(1-\xi)\,\frac{r_2(\xi,\alpha)}{r_0(\xi,\alpha)} \qquad \varepsilon = \frac{1}{N}$$

$$= \int_0^1 d\xi\,(1-\xi)\,\frac{96\alpha^2}{(1+48\xi\alpha)^2} \quad \text{— (b)}$$

- $$\frac{N}{2}(1-\xi)\ln\left(\frac{r_0(\xi,\alpha)}{\xi}\right)\Bigg|_{\xi=0}^{\xi=1} \qquad \left[\frac{r_0(\xi,\alpha)}{\xi} = \frac{-1+\sqrt{1+48\alpha\xi}}{24\alpha\xi}\right]$$

$$= \frac{N}{2}\left\{\underbrace{0\times\ln\left(\frac{-1+\sqrt{1+48\alpha}}{24\alpha}\right)}_{0} - \underbrace{\ln\left(\frac{-1+(1+24\alpha\xi)}{24\alpha\xi}\right)}_{0}\right\} \quad \text{— (c)}$$

- $$\frac{1}{12}\left\{-\ln\frac{r_0(\xi,\alpha)}{\xi} + (1-\xi)\,\partial_\xi\left(\ln\frac{r_0}{\xi}\right)\right\}\Bigg|_{\xi=0}^{\xi=1}$$

$$= \frac{1}{12}\Big[-\Big\{ \ln\Big(\frac{\sqrt{1+48\alpha}-1}{24\alpha}\Big) - \underbrace{\ln(1)}_{0} \Big\} + (1-\xi)\Big\{ \partial_\xi \ln r_0(\xi)\Big|_0^1 - \frac{1}{3}\Big\}\Big] \qquad \text{—(d)}$$

Collecting (a), (b), (c), (d)

we get

$$F^{(1)}(\alpha) = \frac{1}{24}\Big(\ln 4 + \ln(1+48\alpha) - 2\ln\big(1+\sqrt{1+48\alpha}\,\big)\Big)$$

$$\frac{1}{24}\big(\ln 4 + 48\alpha - 2\ln(2+24\alpha)\big) = \frac{1}{24}\big[\ln 4 + 48\alpha - 2\ln 2 - 24\alpha\big] = \alpha + O(\alpha^2)$$

$F^{(1)}(\alpha)$ is $\propto \log(\alpha-\alpha_c)$

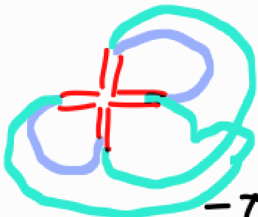
Small α expansion

$$F^{(0)}(\alpha) = 2\alpha - 18\alpha^2 + 288\alpha^3 + O(\alpha^4)$$

$$F^{(1)}(\alpha) = \alpha - 30\alpha^2 + 1056\alpha^3 + O(\alpha^4)$$

Note the $O(\alpha)$ term

$$F(\alpha) = \alpha\Big(2 + \frac{1}{N^2}\Big) \qquad \text{cf.}$$



Higher order $F^{(2)}(\alpha) \propto (\alpha-\alpha_c)^{-5/2}$

Non-Perturbative

From (5) we get (cf. Anninos (4.20))

$$\xi = r(\xi) + 4\alpha\, r(\xi)\big(r(\xi) + r(\xi+\varepsilon) + r(\xi-\varepsilon)\big) \qquad \text{—(5')}$$

$$-\xi + r(\xi) + 4\alpha\, r(\xi)\big(\underbrace{r(\xi)}_{3r(\xi)-2r(\xi)} + r(\xi+\varepsilon) + r(\xi-\varepsilon)\big) = 0$$

$$-\xi + \underbrace{(r + 12\alpha r^2)}_{W(r)} + 4\alpha r(\xi)\big(r(\xi+\varepsilon) + r(\xi-\varepsilon) - 2r(\xi)\big) \quad \text{---} \; (*)$$

$$\boxed{\begin{aligned} &\xi = 1 + \Delta\alpha\, z \\ &r(\xi) = r_c + \sqrt{\Delta\alpha}\,\delta r(z) \\ &r_c \ni W'(r_c) = 0 \\ &1 + 24\alpha r_c = 0 \\ &\Rightarrow r_c = -1/24\alpha \end{aligned}}$$

$$r(\xi+\varepsilon) = r(\xi) + \varepsilon r'(\xi) + \varepsilon^2 r''(\xi)/2$$
$$r(\xi-\varepsilon) = r(\xi) - \varepsilon r'(\xi) + \varepsilon^2 r''(\xi)/2$$

$$r(\xi) = r_c + \sqrt{\Delta\alpha}\,\delta r(z)$$
$$r'(\xi) = \sqrt{\Delta\alpha}\,\frac{\partial}{\partial\xi}\delta r(z) = \sqrt{\Delta\alpha}\cdot\frac{1}{\Delta\alpha}\frac{\partial}{\partial z}\delta r = \frac{\delta r'(z)}{\sqrt{\Delta\alpha}}$$
$$r''(\xi) = \sqrt{\Delta\alpha}\,\frac{\partial^2}{\partial\xi^2}\delta r(z) = \sqrt{\Delta\alpha}\,\frac{1}{\Delta\alpha^2}\delta r''(z) = \frac{1}{(\Delta\alpha)^{3/2}}\delta r''(z)$$

$$-2r(\xi) + r(\xi+\varepsilon) + r(\xi-\varepsilon) = \varepsilon^2 \frac{1}{(\Delta\alpha)^{3/2}}\delta r''(z)$$

$$\boxed{\varepsilon^2 = (\Delta\alpha)^{5/2}\kappa^2}$$
$$\frac{1}{\kappa} = N(\alpha - \alpha_c)^{5/4}$$

$$(*) \Longrightarrow \; -1 - \Delta\alpha\, z + W(r_c,\alpha) + \tfrac{1}{2}W''(r_c,\alpha)\,\Delta\alpha\,\delta r^2$$
$$-\tfrac{1}{6} \quad + 4\alpha r_c \times \frac{\varepsilon^2}{(\Delta\alpha)^{3/2}}\delta r''(z) = 0 \quad \text{---} \; (**)$$

Now, $W(r_c,\alpha) = r_c + 12\alpha r_c^2 = -\frac{1}{24\alpha} + \frac{12\alpha}{(24\alpha)^2} = \frac{-12\alpha}{(24\alpha)^2} = -\frac{1}{48\alpha}$

$= -\frac{1}{48(-\frac{1}{48} + \Delta\alpha)}$

$\simeq 1 + 48\Delta\alpha$ (*)

$\frac{1}{2}W''(r_c,\alpha)\,\Delta\alpha\,\delta r^2 = 12\alpha\,\Delta\alpha\,\delta r^2$

$\simeq 12\alpha_c\,\Delta\alpha\,\delta r^2$

$= -\frac{1}{4}\Delta\alpha\,\delta r^2$

$\therefore (**) \Longrightarrow$

$$\cancel{-1} - \Delta\alpha\, z + \cancel{1} + 48\Delta\alpha \;{?}\; - \tfrac{1}{4}\Delta\alpha\,\delta r^2$$
$$-\tfrac{1}{6}\kappa^2\Delta\alpha\,\delta r'' = 0$$

$$\varepsilon^2 = \kappa^2\Delta\alpha^{5/2}$$

$$z + \tfrac{1}{4}\delta r^2 + \tfrac{1}{6}\kappa^2\delta r'' = 0$$

(*) The prescription seems to be to expand

$$W(r,\alpha) = W(r_c,\alpha) + \tfrac{1}{2}W''(r_c,\alpha)\,\delta r^2 = W(r_c,\alpha_c) + \tfrac{1}{2}W''\Delta\alpha\,\delta r^2$$

$z + \frac{1}{4}\delta r^2 + \frac{1}{6}\kappa^2 \delta r'' = 0$ —(A)

[This is a Painleve (Type I) equation $\frac{d^2y}{dt^2} = 6y^2 + t$ —(B)]

What are the boundary conditions for (A) ?

2nd order $\Rightarrow$ 2 solutions $\delta r_\pm(z)$

expansion are given by

$$\delta r_\pm(z) = (-z)^{1/2}\left(\pm 2 + \frac{1}{12}\frac{\kappa^2}{(-z)^{5/2}} \mp \frac{49}{576}\frac{\kappa^4}{(-z)^5} + \frac{1225}{3456}\frac{\kappa^6}{(-z)^{15/2}} + \sum_{n=4}^{\infty} a_n^{(\pm)}\frac{\kappa^{2n}}{(-z)^{5n/2}}\right). \quad (4.34)$$

$\delta r_+(z)$ oscillatory and is not relevant for us

$\delta r_-(z)$ is the relevant solution

$$\delta r_-(z) = a_0(-z)^{1/2} + a_1\left(\frac{\kappa}{(-z)^{5/4}}\right)^2 + \cdots$$

$a_0 = -2$

$$a_{n+1} = \frac{25n^2-1}{24}a_n + \frac{1}{4}\sum_{m=1}^{n} a_m a_{n+1-m}$$

e.g. $a_1 = -\frac{1}{24}a_0 = \frac{1}{12}$ ✓

$$a_2 = \frac{25-1}{24}a_1 + \frac{1}{4}a_1^2 = \frac{1}{12} + \frac{1}{4\times 12^2} = \frac{48+1}{4\times 12^2}$$
$$= \frac{49}{576} \checkmark$$

$a_n \sim (2n)!$

$\Rightarrow$ divergent series

Such solutions are dealt with by sophisticated methods (see, e.g. Ginsparg - Zinn Justin PLB 255 (1991) 189 ; more generally Bender and Orszag)

Such methods lead to the following non-perturbative behaviours ($\sim e^{-\frac{\#}{\kappa}} \sim e^{-\# N}$)

$$\delta r(z) \sim c\left(\frac{-z}{\kappa^{4/5}}\right)^{-\frac{1}{8}} \exp\left[-\frac{4}{5\kappa}(-z)^{5/4}\right]$$

This is related to single eigenvalue tunneling (David, "Phases of Large N matrix model and nonperturbative effects in 2D gravity", 1990)

Back to partition fn

$$\mathcal{F}_N(\alpha) \simeq -N^2\int_0^1 d\xi(1-\xi)\ln\frac{r(\xi,\alpha)}{\xi}$$

$$\mathcal{F}(\kappa) \equiv \lim_{d.s.l.}\left(\mathcal{F}_N(\alpha) - N^2\mathcal{F}^{(0)}(\alpha)\right)$$

(d.s.l $\Rightarrow$ double scaling limit

$r = r_c + \sqrt{\Delta\alpha}\,\delta r$

$\xi = 1 + \Delta\alpha\, z$

$\varepsilon = \Delta\alpha^{5/4}\kappa$