Representation of Lorentz Group and Dirac fields

Sources → Sydney Colemen Notes

→ Peskin

→ Georgie Lie Groups in particle physics.

→ David Tong Notes

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Review of Group theory :-

Group theory is the study of symmetries in physics. Symmetries of a physical theory are sets of transformations which leaves some properties of the physical theory invariant. Those transformations can be thought of as elements of group. Given a set of transformations $T_i, T_J, \ldots$ if we perform the first transformation on the physical theory T_i; then perform subsequent transformation T_J; the result from both the transformation can be thought of as a transformation T_K which belong to the same set. We write it as $\underbrace{T_J \cdot T_i}_{\text{first } T_i \text{ then } T_J} = T_K$

Thus we want the Group to follow this closure property.

Def^n:- A Group G is a set with a rule for assigning to every (ordered) pair of elements, a third element obeying

(i) If $f, g \in G$ then $h = fg \in G$

(ii) For $f, g, h \in G$ then $f(gh) = (fg)h$

(iii) $\forall f \in G \;\exists e :\; ef = fe = f$

(iv) $\forall f \in G$ there exist an inverse f^{-1} such that $ff^{-1} = f^{-1}f = e$

Thus if a group is discrete the group is a multiplication table specifying $g_1 g_2 \;\forall\; g_1 g_2 \in G$

	e	g_1	g_2	g_3	
e	e	g_1	g_2	g_3	$\ldots$
g_1	g_1	$g_1 g_1$	$g_1 g_2$	$g_1 g_3$	$\ldots$
g_2	g_2	$g_2 g_1$	$g_2 g_2$	$g_2 g_3$	
	$\vdots$	$\vdots$		$\vdots$	$\ldots$

Def^n :- A Representation of G is a mapping D of the elements of G onto a set of linear operators with the following property :—

(i) $D(e) = \mathbb{I} \rightarrow$ identity operator in space on which linear operator act

(ii) $\underbrace{D(g_1)\, D(g_2)}_{\text{natural multiplication in the linear space on which linear operators act.}} = D(\underbrace{g_1 g_2}_{\text{group multiplication}})$

Example :- $\mathbb{Z}_3$ [cyclic group of $\overbrace{\text{order}}^{\text{no. of elements in finite group.}}$ 3]

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

$\mathbb{Z}_3$ is Abelian since for $g_1 g_2 \in G$

$g_1 g_2 = g_2 g_1$

→ One Representation of $\mathbb{Z}_3$ is [1-dimensional Representation]

$$D(e) = 1 \qquad D(a) = e^{2\pi i/3} \qquad D(b) = e^{4\pi i/3}$$

→ There is one another way of representating $\mathbb{Z}_3$. The trick is to take group elements and form an orthonormal basis for a vector space $|e\rangle$, $|a\rangle$ and $|b\rangle$. Now we define :-

$$D(g_1)\,|g_2\rangle = |g_1 g_2\rangle$$

This is indeed a representation called the regular representation. lets find the regular representation corresponding to

$$D(d\alpha) = 1 + i\, d\alpha_a X_a + \dots$$ where

$$\boxed{X_a = -i \frac{\partial}{\partial \alpha_a} D(\alpha)\Big|_{\alpha=0}}$$

This are called generators of Group

Now if we go away from the identity in some fixed direction we can just raise the infinitesimal group element

$$D(\alpha) = \lim_{k \to \infty} \Big(1 + i \underbrace{d\alpha_a}_{\frac{\alpha_a}{k}} X_a\Big)^k = e^{i\alpha_a X_a}$$

Thus this mean that we can write group elements in terms of the generators.

However if we multiply two group elements generated by two different linear combination of generators then

$$e^{i\alpha_a X_a}\, e^{i\beta_b X_b} \neq e^{i(\alpha_a + \beta_a) X_a}$$

But the product in the representation should be some exponential of generator

$$\therefore\; e^{i\alpha_a X_a}\, e^{i\beta_b X_b} = e^{i\delta_a X_a}$$

We shall now expand both side and equate powers of α and β.

let us check leading order

$$i\,\delta_a X_a = \ln\big[1 + \underbrace{e^{i\alpha_a X_a}\, e^{i\beta_b X_b} - 1}_{K}\big] \rightarrow 1$$

$$k = e^{i\alpha_a X_a} e^{i\beta_b X_b} - 1$$

$$= \left(1 + i\alpha_a X_a - \frac{1}{2}(\alpha_a X_a)^2 + \cdots\right)\left(1 + i\beta_b X_b - \frac{1}{2}(\beta_b X_b)^2 + \cdots\right) - 1$$

$$= i\alpha_a X_a + i\beta_b X_b - \frac{1}{2}(\alpha_a X_a)^2 - \frac{1}{2}(\beta_b X_b)^2 - \alpha_a X_a \beta_b X_b$$

Now

$$i\delta_a X_a = k - \frac{1}{2}k^2$$

$$= i\alpha_a X_a + i\beta_b X_b - \frac{1}{2}(\alpha_a X_a)^2 - \frac{1}{2}(\beta_b X_b)^2 - \alpha_a X_a \beta_b X_b + \frac{1}{2}(\alpha_a X_a + \beta_b X_b)^2$$

$$= i\alpha_a X_a + i\beta_a X_a - \frac{1}{2}[\alpha_a X_a, \beta_b X_b]$$

$$\Rightarrow [\alpha_a X_a \; \beta_b X_b] = \overbrace{-2i(\delta_c - \alpha_c - \beta_c)X_c + \underbrace{\cdots}}^{\text{The whole thing is } i\gamma_c X_c}$$

represent terms that have more than two factors of α or β

Let say $a = \{1, 2\}$ then

$$[\alpha_1 X_1 + \alpha_2 X_2, \beta_1 X_1 + \beta_2 X_2]$$

$$= [\alpha_1 X_1, \beta_1 X_1] + [\alpha_1 X_1 + \beta_2 X_2] + [\alpha_2 X_2, \beta_1 X_1] + [\alpha_2 X_2, \beta_2 X_2]$$

$$= \alpha_1\beta_1 \cancel{[X_1, X_1]} + \alpha_1\beta_2 [X_1 X_2] + \alpha_2\beta_1 [X_2 X_1] + \alpha_2\beta_2 \cancel{[X_2 X_2]}$$

$$= \alpha_1\beta_2 [X_1 X_2] + \alpha_2\beta_1 [X_2 X_1]$$

which can be generalized as $\alpha_a \beta_b [X_a, X_b]$

The right hand side can be defined as $i \gamma_c X_c =$

where $\gamma_c = -2(\delta_c - \alpha_c - \beta_c)$

Thus we can define some constants f_{abc} for which

$$\boxed{[X_a, X_b] = i f_{abc} X_c}$$

Generators form an algebra under commutation

Exchanging a and b we get

$$[X_b, X_a] = i f_{bac} X_c$$

$$\Rightarrow -[X_a, X_b] = i f_{bac} X_c$$

$$\Rightarrow \boxed{f_{abc} = -f_{bac}}$$

these are called structure constant

su(2) algebra is familiar

$$[J_j, J_k] = i \epsilon_{jkl} J_l$$

Lets now move towards SO(3,1) group which is the group of orthogonal transformations with determinant 1 which preserves the square of the Minkowski norm

$$x_0^2 - x_1^2 - x_2^2 - x_3^2.$$

let us now look at the transformation of fields under Lorentz group. For a scalar field, the transformation law is given as

$$\phi \longrightarrow \phi(\Lambda^{-1} x)$$

For a vector field A_μ, the transformation law is

$$A_\mu \rightarrow \Lambda_\mu{}^\nu A_\nu(\Lambda^{-1} x)$$

The above fields describe elements with integer spins. But if we want to describe transformation law of half-integer spins we can write a general law for fields

$$\boxed{\phi_a(x) \longrightarrow \underbrace{D_{ab}(\Lambda)}\, \phi_b(\Lambda^{-1}x)}$$

Representation of Lorentz transformation.

This can be more complicated matrix depending on what sort fields we are describing. For most of our theory, we shall be needing the fields that describes spin $1/2$ particles which are electrons, protons etc..

The Elements of the Lorentz group Λ has certain properties which needs to be satisfied by the representation

1. If $\Lambda_1, \Lambda_2 \in \Lambda$ then

$$\Lambda_1 \Lambda_2 = \Lambda_3 \in \Lambda \text{ , then we have}$$

$$D(\Lambda_1)\, D(\Lambda_2) = D(\Lambda_1 \Lambda_2) = D(\Lambda_3)$$

Now for Λ and Λ^{-1} we have.

$$D(\Lambda)\, D(\Lambda^{-1}) = D(\Lambda \Lambda^{-1}) = \underbrace{D(\mathbb{I}) = \mathbb{I}}_{\text{first property of the representation}}$$

$$\Rightarrow \boxed{D(\Lambda^{-1}) = [D(\Lambda)]^{-1}}$$

Thus the representation D forms a finite dimensional representation of Lorentz group.

Let us suppose $D(\Lambda)$ is a representation then

$$D(\Lambda)' = T\, D(\Lambda)\, T^{-1} \text{ for any fixed } T \text{ is also a representation}$$

to prove this

$$D(\Lambda_1)' D(\Lambda_2)' = T D(\Lambda_1) T^{-1} T D(\Lambda_2) T^{-1}$$

$$= T D(\Lambda_1) D(\Lambda_2) T^{-1}$$

$$= T D(\Lambda_1 \Lambda_2) T^{-1}$$

$$= D'(\Lambda_1 \Lambda_2)$$

→ which satisfies multiplicative law of representation.

If two representations are related in this way then we say

$D(\Lambda) \sim D'(\Lambda)$ (equivalent)

Thus we can see that given a representation we can always perform similarity transformation to get a different representation which are equivalent to previous one. There is one more way of generating representation from the old one. Suppose $D^{1}(\Lambda)$ and $D^2(\Lambda)$ of dim n_1 and dim n_2 are two representations, then we can make.

$$D(\Lambda) = \begin{bmatrix} D^{(1)}(\Lambda) & \\ & D^{(2)}(\Lambda) \end{bmatrix} \equiv D^{(1)}(\Lambda) \oplus D^{(2)}(\Lambda)$$

This too is a representation

Direct Sum

with $\dim D(\Lambda) = \dim D^{(1)}(\Lambda) + \dim^{(2)}(\Lambda)$

But we are not interested in representations that can be written (reduced) into direct sum. We call such representation "reducible". So our task will be to find all the irreducible finite dimensional representation of the Lorentz Group. lets first compute the irreducible representation of a subgroup which is Rotation group $SO(3)$.

group of rotation in space R^3 about some axis by some angle.

The rotation matrix R can be labelled by an axis $\vec{n}$ and some angle θ

$$R \in SO(3) \; : \; R(\vec{n}\,\theta) = R \qquad 0 \leq \theta \leq \pi$$

We observe that

$$R(\vec{n}\,\theta)\, R(\hat{n}\,\theta') = R(\vec{n}(\theta+\theta'))$$. Thus the representations will also satisfy

$$D(R(\vec{n}\,\theta))\, D(R(\vec{n}\,\theta')) = D(R(\vec{n}(\theta+\theta')))$$

Take the derivative at $\theta' = 0$

$$D(R(\vec{n}\,\theta)) \left.\frac{\partial}{\partial \theta'} D(R(\vec{n}\,\theta'))\right|_{\theta'=0} = \left.\frac{\partial}{\partial(\theta+\theta')} D(R(\vec{n}(\theta+\theta')))\right|_{\theta'=0}$$

We shall define

$$\left.\frac{\partial}{\partial \theta} D(R(\vec{n}\,\theta))\right|_{\theta=0} = -i\vec{n}\cdot\mathbb{L}$$

$$-i\vec{n}\cdot\mathbb{L}\, D(R(\vec{n}\,\theta)) = \frac{\partial}{\partial\theta} D(R(\vec{n}\,\theta))$$

$$\Rightarrow \boxed{D(R(\vec{n}\,\theta)) = e^{-i\vec{n}\cdot\underbrace{\vec{\mathbb{L}}}\theta}}$$ By putting $D(R(\theta=0)) = 1$

generators of the Lorentz group $\{L^i\}$ where $i = 1,2,3$

<u>Working out the algebra of the matrices $\{L^i\}$</u>

The transformation of vector $\vec{v}$ about any axis $\vec{n}$ by an infinitesimal rotation by θ is given by

$$v \longrightarrow v + \theta\,\vec{n}\times\vec{v} + O(v^2)$$

Now the generators $\{L^i\}$ of the group act as an operator in the linear space of the group elements. We shall look at how generatoors transforms under rotation. Lets look for a general operator A. It acts on state $|\psi\rangle$ as

$$A|\psi\rangle = |\phi\rangle$$

$$\Rightarrow A\, D(R)^{\dagger} D(R)\, |\psi\rangle = |\phi\rangle$$

$$\Rightarrow \underbrace{D(R)\, A\, D(R)^{\dagger}}_{A'} \underbrace{D(R)\, |\psi\rangle}_{|\psi'\rangle} = D(R)\, |\phi\rangle = |\phi'\rangle$$

$$\therefore \boxed{A' = D(R)\, A\, D(R)^{\dagger}}$$

any operator transforms like this

Thus for infinitesimal transformation we have

$$\left(1 - i\hat{n}\theta\cdot\mathbb{L}\right) A \left(1 + i\hat{n}\theta\cdot\mathbb{L}\right) = A'$$

$$\Rightarrow A + A\, i\hat{n}\,\theta\cdot L - i\hat{n}\,\theta\cdot\mathbb{L}\, A = A'$$

$$\Rightarrow A' = A + i\theta n_k \left[L_k, A\right]$$

Now when A is rotationally invariant, then $A' = A$

$$\Rightarrow \boxed{\left[L_k, A\right] = 0}$$

For a velocity vector we have

$$\vec{v}' = \vec{v} + i\theta n_k \left[L_k, \vec{v}\right]$$

$$\Rightarrow \cancel{v_i} + \varepsilon_{ijk}\, \theta n_j v_k = \cancel{v_i} + i\,\theta n_k \left[L_k, v_i\right]$$

$$\Rightarrow \left[L_k, v_i\right] = -i\,\varepsilon_{ikj}\, v_j$$

$$\Rightarrow \boxed{\left[L_i, v_j\right] = i\varepsilon_{ijk}\, v_k}$$

Since $\{\mathbb{L}\}$ also form a vector in the linear space we shall have the algebra $\boxed{[L_i, L_j] = i\epsilon_{ijk} L_k}$ } The famous angular momentum commutator.

Finding this generators will get us the representation. Thus if we can find up to equivalence and direct sum, all matrices that obey these commutation relations we shall have all the rep of the Rotation group.

Finite Dimensional inequivalent irrep of the Lie algebra of Rotation group are notated by $D^{(s)}(R)$ labelled by an index "s".

$$D^{(s)}(R) = e^{-i\,\hat{n}\theta\cdot \underbrace{\mathbb{L}^{(s)}}}$$

triplet of matrices appropriate to spin s.

We have

$\mathbb{L}^{(s)}$ $\quad s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

$\vec{L}^{(0)} = \vec{0}$

$\vec{\mathbb{L}}^{(1/2)} = \dfrac{\vec{\sigma}}{2}$ where σ = Pauli matrices.

→ The dimension of the representation $D^{(s)}(R)$ is $2s+1$

→ The square of $\mathbb{L}^{(s)}$ is multiple of identity

$$\mathbb{L}^{(s)} \cdot \mathbb{L}^{(s)} = s(s+1)\mathbb{I}$$

If we choose one component of $\mathbb{L}^{(s)}$ lets say $\mathbb{L}_z^{(s)}$ then we shall have $\mathbb{L}_z^{(s)} |m\rangle = m|m\rangle$ where

$$m = -s, -s+1, -s+2, \ldots, s-2, s-1, s$$

Some facts :- ① The representation of lie algebra just listed above not only generates the representation of Rotation group they generate representation upto a phase. The integer s are representation. the half integer s are reps upto a phase. ie they are double valued

$$D^{(s)}\left(R(2\pi \hat{n})\right) = (-1)^{2s} \mathbb{1}$$

② If $D^{(s)}(R)$ is a rep of $SO(3)$ then so is $D^{(s)}(R)^*$.

$$\therefore \quad D^{(s)}(R) \sim D^{(s)}(R)^*$$

③ If we have some sets of fields that transforms under rotation as a irrep $D^{(s_1)}(R)$ and second sets of fields that transforms as another irrep $D^{(s_2)}(R)$ then we can get a new representation given by

$$D^{(s_1)}(R) \otimes D^{(s_2)}(R)$$

But its not necessary a irreducible representation

The dim of the direct product is

$$(2s_1+1)(2s_2+1)$$

There is a rule of how we can break it up into irreducible representations. It is equivalent to direct sum which can be indicated as

$$D^{(s_1)}(R) \otimes D^{(s_2)}(R) \sim \oplus \sum_{s=|s_1-s_2|}^{s_1+s_2} D^{(s)}(R)$$

For eg

$$D^{(1/2)}(R) \otimes D^{(1/2)}(R) \sim D^{(0)} \oplus D^{(1)}$$

$\underbrace{\hspace{5em}}$ The product of spinors give two object $\underbrace{\hspace{5em}}$ a scalar and a vector.

Lorentz Group

Lorentz transformation can be decomposed into a rotation and a boost. A boost $A(\hat{a}\phi)$ along a given axis $\vec{a}$ and rapidity ϕ is a pure Lorentz transformation that takes a particle at rest and changes its $\vec{\text{velocity}}$ to some new value along that axis.

As with rotations, we have

$$A(\hat{a}\,\phi)\,A(\hat{a}\,\phi') = A(\hat{a}\,(\phi+\phi'))$$

By defining

$$-i\,\hat{a}\,\phi\cdot \underset{\sim}{M} = \left.\frac{\partial D(A(\vec{a}\,\phi))}{\partial\phi}\right|_{\phi=0}$$

M is the generator of boosts.

and we shall find that

$$D(A(\hat{a}\,\phi)) = e^{-i\,\vec{a}\cdot M\phi}$$

Thus if we know L and M we know the representation matrix for arbitrary rotation and arbitrary boosts and by multiplication we can find the representation matrix for any general Lorentz transformation. Let us now write all the commutators of L and M. For rotations we have.

$$[L_i, L_j] = i\epsilon_{ijk}L_k$$

Next $$[L_i, M_j] = i\epsilon_{ijk}M_k$$ } This tells us that M transforms like a vector.

Now $$[M_i, M_j] = -i\,\epsilon_{ijk} L_k$$

The minus sign here is very important.

Now to find all the irreducible representation of Lorentz algebra,

We shall now use a trick:- We shall now define

$$J^{\pm} = \frac{1}{2}(L \pm iM) \quad \text{so we have}$$

$$L = J^{+} + J^{-}$$

$$M = -i\,(J^{+} - J^{-})$$

Let us compute the commutation of these new operators and we shall see

$$[J_i^{(-)}, J_j^{(-)}] = i\,\epsilon_{ijk} J_k^{(-)}$$

$$[J_i^{(+)}, J_j^{(+)}] = i\,\epsilon_{ijk} J_k^{(+)}$$

$$[J_i^{(+)}, J_j^{(-)}] = 0$$

Thus $\{J_i^{(+)}\}$ and $\{J_j^{(-)}\}$ commute with each other

The two $\{J_i^{(+)}\}$ and $\{J_j^{(-)}\}$ forms two commuting independent $SU(2)$ algebras. Thus a complete set of irreducible representation of Lorentz group are characterised by two spin quantum no $s+$ and $s-$ one for each J^{+} and J^{-} and written as

$$D^{(s+,\,s-)}(\Lambda) \qquad s\pm = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots$$

The square of these operators J^{+} and J^{-} are multiples of identity

$$J^{+} \cdot J^{+} = s^{+} (s^{+}+1) J^{+}$$

$$J^{-} \cdot J^{-} = s^{-} (s^{-}+1) J^{-}$$

The complete set of basis is defined by two numbers m_+ and m_- which are the eigenvalue of J_z^{+} and J_z^{-} respectively such that

$$J_z^{\pm} \, |m_+ m_-\rangle = m_{\pm} \, |m_+ m_-\rangle$$

These states are simultaneous eigenstates of commuting operators.

We can always choose our basis such that J^{+} and J^{-} are hermitian matrices and so we can see that L is hermitian but not M $\therefore$ $D(R)$ are unitary but $D(A)$ are not.

Properties of $SO(3,1)$ representation $D^{(s_+, s_-)}(\Lambda)$

$$\left[D^{(s_+, s_-)}(\Lambda) \right]^{*} \sim D^{(s_-, s_+)}(\Lambda)$$

$$P : D^{(s_+, s_-)}(\Lambda) \to D^{(s_-, s_+)}(\Lambda)$$

Parity

This is coz parity turns L into L and M into $-M$. The operation $M \to -M$ can be thought of as exchanging $J^{(+)}$ and $J^{(-)}$

$$D^{(s_+, s_-)}(R) \sim \bigoplus_{s=|s_+ - s_-|}^{s_+ + s_-} D^{(s)}(R)$$

Extras:- Finding the general commutations of the generators of Lorentz group!

We know in quantum mechanics generators of rotation

$$J = x \times p = x \times (-i\nabla)$$

lets write the operators as an antisymmetric tensor

$$J^{ij} = -i\,(x^i \nabla^j - x^j \nabla^i)$$

where $J^3 = J^{12}$ and so on.. The generalization to 4-vector is

$$\boxed{J^{\mu\nu} = i\,(x^\mu \partial^\nu - x^\nu \partial^\mu)}$$

We shall be able to compute now that (A)

$$\boxed{[J^{\mu\nu}, J^{\rho\sigma}] = i\,(\eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho} + \eta^{\mu\sigma} J^{\nu\rho})}$$

lorentz

Any matrices that are to represent this algebra can be used to find the representation of lorentz group.

Lets find the representation corresponding to spin 1/2. Lets use a trick used by Dirac! If we have $n \times n$ matrices γ^μ satisfying the anti-commutation relation which is

Dirac matrices

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \times \mathbb{1}_{n\times n}}$$

(Dirac algebra)

then we could write down an n-dimensional representation of Lorentz algebra. which is

$$\boxed{S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]}$$

We can actually show that this $S^{\mu\nu}$ satisfy (A). This trick can be used for any dimensionality whether Lorentz or Euclidean metric. Lets work it in 3-dimensional Euclidean space where we choose

$$\gamma^j \equiv i\sigma^j \quad \text{(Pauli sigma matrices)}$$

Thus
$$\{\gamma^i, \gamma^j\} = \{i\sigma^i, i\sigma^j\} = -\{\sigma^i, \sigma^j\}$$
$$= -2\delta_{ij}\,\mathbb{I}$$

This minus sign is conventional.

The representation will then be

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j]$$
$$= \frac{i}{4}(-1)[\sigma^i, \sigma^j]$$
$$= \frac{-i}{4}\, 2i\,\epsilon^{ijk}\sigma^k$$
$$\underbrace{S^{ij} = \frac{1}{2}\epsilon^{ijk}\sigma^k}$$

This is the 2-dimensional representation of rotation group.

where we have used

$$\{\sigma^i, \sigma^j\} = 2i\epsilon^{ijk}\sigma^k$$
$$[\sigma^i, \sigma^j] = 2\delta^{ij}\,\mathbb{I}$$

We shall need to find Dirac matrices real for Minkowski space

one representations in 2×2 block form is

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

This representation is called Weyl or Chiral representation

Thus using

$$S^{\mu\nu} = \frac{i}{4}\left[\gamma^{\mu}, \gamma^{\nu}\right]$$

Thus

$$S^{0i} = \frac{i}{4}\left[\gamma^{0}, \gamma^{i}\right]$$

$$= \frac{i}{4}\left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^{i} \\ -\sigma^{i} & 0 \end{bmatrix} - \begin{bmatrix} 0 & \sigma^{i} \\ -\sigma^{i} & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

$$= \frac{i}{4}\left\{ \begin{bmatrix} -\sigma^{i} & 0 \\ 0 & \sigma^{i} \end{bmatrix} - \begin{bmatrix} \sigma^{i} & 0 \\ 0 & -\sigma^{i} \end{bmatrix} \right\}$$

$$\boxed{S^{0i} = -\frac{i}{2}\begin{pmatrix} \sigma^{i} & 0 \\ 0 & -\sigma^{i} \end{pmatrix}}$$

This is not hermitian and thus transformation of boosts is not unitary.

Also we can calculate and will find that

$$(S^{0i})^{\dagger} = -S^{0i}$$

$$S^{ij} = \frac{i}{4}\left[\gamma^{i}, \gamma^{j}\right]$$

$$= \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^{k} & 0 \\ 0 & \sigma^{k} \end{pmatrix}$$

$$\boxed{S^{ij} \equiv \frac{1}{2}\epsilon^{ijk}\Sigma^{k}}$$

These are the generators of the Lorentz group and the four component field ψ that transforms under boost and rotation a/c to these generators are called Dirac Spinors. The transformation for the Dirac spinor ψ is given by

$$\boxed{\psi(x) \longmapsto \exp\left(-\frac{i}{2}\Omega_{\mu\nu} S^{\mu\nu}\right) \psi(\Lambda^{-1}x)}$$

Generators

infinitesimal LT

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \Omega^{\mu}{}_{\nu}$$

This acts like the representation matrix of Lorentz group for Dirac spinors.

Next step after giving a field is what the dynamics is and for that we need a Lagrangian and so the next question is what will be the Lagrangian for Dirac spinors.

How does the field ψ evolve?

<u>Recap of scalars and Vectors</u>

Scalar :- We know that $\phi(x) \longmapsto \phi(\Lambda^{-1}x)$. So the terms permissible in the Lagrangian

$$\int d^4x\, \phi^2(x)$$

$$\downarrow$$

$$\int d^4x\, \phi(\Lambda^{-1}x)\phi(\Lambda^{-1}x)$$

When you change $y = \Lambda^{-1}x$

$$\underbrace{\int d^4y}\, \phi(y)\,\phi(y)$$

Under Lorentz Transformation, the measure donot change.

Lets now consider $\partial_\mu\phi$ coz we want some dynamics of ϕ, if I just wrote $\phi^2(x)$ Everything will just be the same. In space time I want things to change. How does $\partial_\mu\phi$ transforms as

$$\partial_\mu\phi \longmapsto \Lambda_\mu{}^{\nu}\, \phi(\Lambda^{-1}x)$$

Thus

$$\int \partial_\mu\phi\, \partial^\mu\phi\, d^4x \longmapsto \int \Lambda_\mu{}^{\nu}\, \Lambda^{\mu}{}_{\nu}\, \phi(\Lambda^{-1}x)\, \phi(\Lambda^{-1}x)\, d^4x$$

$$= \int \partial_\mu \phi \, \partial^\mu \phi \, d^4 y$$

So the simplest lagrangian that you can construct is

$$S_{\text{free scalar}} = \frac{1}{2} \int d^4x \left(\partial_\mu \phi \, \partial^\mu \phi - m^2 \phi^2 \right)$$

This is just a number outside.

lets now come to vectors $\longrightarrow$ (Maxwell)

$$A_\mu \xmapsto{LT} \Lambda_\mu{}^\nu \partial_\mu A(\Lambda^{-1} x)$$

But you also say that I want a gauge invariant theory on top of Lorentz invariance.

$$A_\mu \xmapsto{GT} A_\mu + \partial_\mu \Lambda$$

$\therefore$ Gauge invariance forces us to consider

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \text{ such that}$$

$$F_{\mu\nu} \xrightarrow{GT} F_{\mu\nu}$$

and so you run the same argument again and the simplest action that you can make is

$$S_{\text{maxwell}} = -\frac{1}{4} \int d^4x \, F_{\mu\nu} F^{\mu\nu}$$

Now we want to do the same procedure for Ψ. For fermions we have $S^{\mu\nu}$, $\Omega_{\mu\nu}$ and exponentials so its not that trivial we have to do some work. Lets consider an infinitesimal LT

$$\boxed{\psi(x) \longmapsto \left(\mathbb{1}_{4\times 4} - \frac{i}{2} \Omega_{\mu\nu} S^{\mu\nu} + \cdots \right) \psi(\Lambda^{-1} x)}$$

lets try contructing a scalar out of this. ϕ was a real scalar and for complex scalar we would have done $\phi^* \phi$ and since Ψ is complex so lets check. $\psi^* \psi$

$$\psi^*(x) \longmapsto \psi^+(\Lambda^{-1}x)\left(\mathbb{1}_{4\times 4} + \frac{i}{2}\Omega_{\mu\nu}\underbrace{(S^{\mu\nu})^+}\right)$$

This is neither hermitian or anti-hermitian

So

$$\psi^+\psi = \psi^+(\Lambda^{-1}x)\left(\mathbb{1}_{4\times 4} + \frac{i}{2}\Omega_{\mu\nu}(S^{\mu\nu})^+ + \dots\right)$$

$$\left(\mathbb{1}_{4\times 4} - \frac{i}{2}\Omega_{\mu\nu}S^{\mu\nu} + \dots\right)\Psi(\Lambda^{-1}x)$$

$$= \psi^+(\Lambda^{-1}x)\left(\mathbb{1}_{4\times 4} - \frac{i}{2}\Omega_{\mu\nu}S^{\mu\nu} + \frac{i}{2}\Omega_{\mu\nu}(S^{\mu\nu})^+ + \dots\right)\Psi(\Lambda^{-1}x)$$

Now $S^{\mu\nu}$ is not hermitian and so we can cancel the two terms and so $\psi^+\psi$ is not scalar.

At this moment you must have a stroke of genius and consider

$$\overline{\Psi} = \psi^+\gamma^0$$

$$\overline{\Psi}(x) \longmapsto \psi^+(\Lambda^{-1}x)\left(\mathbb{1}_{4\times 4} + \frac{i}{2}\Omega_{\mu\nu}(S^{\mu\nu})^+ + \dots\right)\gamma^0$$

$$= \psi^+(\Lambda^{-1}x)\left(\mathbb{1}_{4\times 4} + \frac{i}{2}\times 2\Omega_{0i}\underbrace{S^{0i}} - \frac{i}{2}\times 2\Omega_{ij}S^{ij} + \dots\right)\gamma^0$$

$$(S^{0i})^\dagger = S^{0i}$$

Now lets see how γ^0 commutes with S^{0i} and S^{ij}.

Claim:– γ^0 commutes with S^{ij} and anti-commutes with S^{0i}

Thus

$$\bar{\Psi}(x') \longmapsto \underbrace{\psi^\dagger(\Lambda^{-1}x)\,\gamma^0}\left(\mathbb{1}_{4\times4} - \frac{i}{2}\times 2\,\Omega_{0i}\,S^{0i} - \frac{i}{2}\times 2\,\Omega_{ij}\,S^{i}{}_{j} + \cdots\right)$$

$$\bar{\Psi}(\Lambda^{-1}x)\left(\mathbb{1}_{4\times4} - \frac{i}{2}\Omega_{\mu\nu}S^{\mu\nu}\right)$$

and we have

$$\Psi(x) \longmapsto \left(\mathbb{1}_{4\times4} + \frac{i}{2}\Omega_{\mu\nu}S^{\mu\nu}\right)\Psi(\Lambda^{-1}x)$$. Therefore

we have

$$\boxed{\int d^4x\, \bar{\Psi}\,\Psi \text{ is a scalar}}$$

This is admissible term in the Lagrangian analogous to ϕ^2 in the scalar field theory.

Proving the Claim:–

$$\{\gamma^0, S^{0i}\} = \frac{-i}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} + \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$$

$$= \frac{-i}{2}\left[\begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\right]$$

$= 0$ Thus γ^0 anti-commutes with S^{0i}

Now lets check

$$[\gamma^0, S^{ij}] = \frac{1}{2}\epsilon^{ijk}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} - \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$$

$$= \frac{1}{2}\epsilon^{ijk}\left[\begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}\right]$$

$$= 0$$

γ^0 commutes with S^{ij}.

Proved

The next step is to incoorporate some dynamics on the Dirac spinor ψ in the Lagrangian. I want to write down the simplest form of derivative in the Lagrangian. We want

$$\bar{\psi} \underbrace{\gamma^\mu}\, \partial_\mu \psi$$

To have some indices up and I can do that with γ^μ and to make it same as $\bar{\psi}\psi$ just put $\bar{\psi}$ coz we have learnt have we have to put $\bar{\psi}$ to make it a scalar.

$$\bar{\psi}\gamma^\mu\partial_\mu\psi \longmapsto \bar{\psi}\left(\mathbb{1}_{4\times4} - \frac{i}{2}\Omega_{\alpha\beta}S^{\alpha\beta}\right)\gamma^\mu$$

$$\times \underbrace{\Lambda_\mu{}^\nu}_{(\delta_\mu{}^\nu + \Omega_\mu{}^\nu)}\partial_\nu\left(\mathbb{1}_{4\times4} + \frac{i}{2}\Omega_{\gamma\delta}S^{\gamma\delta}\right)\psi$$

$$= \bar{\psi} \Big\{ \mathbb{1}_{4\times4} \gamma^\mu \partial_\mu - \frac{i}{2} \Omega_{\alpha\beta} S^{\alpha\beta} \gamma^\mu \partial_\mu$$

$$+ \mathbb{1} \gamma^\mu \underbrace{\Omega_\mu{}^\nu \partial_\nu}_{\Omega_{\mu\nu}\partial^\nu} + \frac{i}{2} \gamma^\mu \partial_\mu \Omega_{\gamma\delta} S^{\gamma\delta} \Big\} \psi$$

$$= \bar{\psi} \gamma^\mu \partial_\mu \psi + \text{(rest should cancel)}$$

$$= \bar{\psi} \gamma^\mu \partial_\mu \psi + \bar{\psi} \gamma^\mu \Omega_\mu{}^\nu \partial_\nu \psi + \frac{i}{2} \bar{\psi} \Omega_{\alpha\beta} [\gamma^\mu, S^{\alpha\beta}] \partial_\mu \psi$$

lets now calculate

$$[\gamma^\mu, S^{\alpha\beta}] = \frac{i}{4} \Big([\gamma^\mu, \gamma^\alpha \gamma^\beta] - [\gamma^\mu, \gamma^\beta \gamma^\alpha] \Big)$$

We know that γ^μ, γ^ν are anti-commutator. Now

$$[A, BC] = \underbrace{\{A, B\} C - B \{A, C\}}_{\text{coz}}$$

$$ABC + \cancel{BAC} - \cancel{BAC} - BCA$$

$$= ABC - BCA = [A, BC]$$

$$\therefore [\gamma^\mu, S^{\alpha\beta}] = \frac{i}{4} \Big(\{\gamma^\mu, \gamma^\alpha\} \gamma^\beta - \gamma^\alpha \{\gamma^\mu, \gamma^\beta\}$$

$$- \{\gamma^\mu, \gamma^\beta\} \gamma^\alpha + \gamma^\beta \{\gamma^\mu, \gamma^\alpha\} \Big)$$

$$= \frac{i}{4} \Big(2 \eta^{\mu\alpha} \gamma^\beta - \gamma^\alpha 2 \eta^{\mu\beta}$$

$$- 2 \eta^{\mu\beta} \gamma^\alpha + 2 \gamma^\beta \eta^{\mu\alpha} \Big)$$

$$= \frac{i}{2}\left(\eta^{\mu\alpha}\gamma^\beta - \gamma^\alpha\eta^{\mu\beta} - \eta^{\mu\beta}\gamma^\alpha + \gamma^\beta\eta^{\mu\alpha}\right)$$

$$\boxed{[\gamma^\mu, S^{\alpha\beta}] = i\left(\eta^{\mu\alpha}\gamma^\beta - \gamma^\alpha\eta^{\mu\beta}\right)}$$

Thus

$$\bar\psi\gamma^\mu\partial_\mu\psi \longmapsto \bar\psi\gamma^\mu\partial_\mu\psi + \bar\psi\gamma^\mu\Omega_{\mu\nu}\partial^\nu\psi + \frac{i}{2}\Omega_{\alpha\beta}\bar\psi\, i\left(\eta^{\mu\alpha}\gamma^\beta - \gamma^\alpha\eta^{\mu\beta}\right)\partial_\mu\psi$$

$$= \bar\psi\gamma^\mu\partial_\mu\psi + \bar\psi\gamma^\mu\Omega_{\mu\nu}\partial^\nu\psi - \frac{\bar\psi}{2}\Omega_{\alpha\beta}\left(\gamma^\beta\partial^\alpha - \gamma^\alpha\partial^\beta\right)\psi$$

$$= \bar\psi\gamma^\mu\partial_\mu\psi + \gamma^\mu\bar\psi\Omega_{\mu\nu}\partial^\nu\psi - \frac{\bar\psi}{2}\underbrace{\underbrace{\left(\Omega_{\alpha\beta}\gamma^\beta\partial^\alpha - \Omega_{\alpha\beta}\gamma^\alpha\partial^\beta\right)\psi}_{\left(\Omega_{\alpha\beta}\gamma^\beta - \Omega_{\beta\alpha}\gamma^\beta\right)\partial^\alpha\psi}}_{-2\Omega_{\beta\alpha}\gamma^\beta\partial^\alpha\psi}$$

$$= \bar\psi\gamma^\mu\partial_\mu\psi + \bar\psi\Omega_{\mu\nu}\gamma^\mu\partial^\nu\psi - \bar\psi\Omega_{\nu\mu}\gamma^\mu\partial^\nu\psi$$

$$= \bar\psi\gamma^\mu\partial_\mu\psi$$

$\therefore$ $\bar\psi\gamma^\mu\partial_\mu\psi$ is a scalar and we can put this in Lagrangian

At the end of the day, the Lagrangian should be scalar

Lets now consider

$$(i\bar{\psi}\gamma^\mu\partial_\mu\psi)^* = (\partial_\mu\psi)^\dagger(\gamma^\mu)^\dagger(-i)(\bar{\psi})^\dagger$$

$(\gamma^0)^\dagger = \gamma^0$

$(\gamma^i)^\dagger = -\gamma^i$

$$= -i(\partial_\mu\psi)^\dagger(\gamma^\mu)^\dagger(\psi^\dagger\gamma^0)^\dagger$$

$$= -i(\partial_\mu\psi)^\dagger(\gamma^\mu)^\dagger\gamma^0\psi$$

$$= -i\{(\partial_0\psi)^\dagger(\gamma^0)^\dagger\gamma^0\psi + (\partial_i\psi)^\dagger(\gamma^i)^\dagger\gamma^0\psi\}$$

$\bar{\psi} = \psi^\dagger\gamma^0$

$$= -i\{(\partial_0\psi\gamma^0)^\dagger\gamma^0\psi - (\partial_i\psi\gamma^0)^\dagger\gamma^i\psi\}$$

$$= -i\{\partial_0\bar{\psi}\gamma^0\psi - \partial_i\bar{\psi}\gamma^i\psi\}$$

$$= -i\,\partial_\mu\bar{\psi}\gamma^\mu\psi$$

We need to show that this is real which means $(i\bar{\psi}\gamma^\mu\partial_\mu\psi)^* \overset{??}{=} i\bar{\psi}\gamma^\mu\partial_\mu\psi$

which we didnot got

But all this terms will go inside the integral and we can do integration by parts

$$\partial_\mu(-i\bar{\psi}\gamma^\mu\psi) + i\bar{\psi}\gamma^\mu\partial_\mu\psi$$

ignoring the total derivative coz the fields are such that γ dies off at infinity

$$(\bar{\psi}\,i\gamma^\mu\partial_\mu\psi)^* = \bar{\psi}\,i\gamma^\mu\partial_\mu\psi$$

$\therefore$ We can write the action as

$$\boxed{S_{Dirac} = \int d^4x\,\bar{\psi}\,(i\gamma^\mu\partial_\mu - m)\,\psi}$$

a constant

This is the action that describes electrons, protons etc. The interpretation that m is a mass will be explained later

The EOL for $\bar{\psi}$ (or $\psi^\dagger$) immediately yields the Dirac equation and we have

$$\boxed{-i\partial_\mu \bar{\psi}\gamma^\mu - m\bar{\psi} = 0} \rightarrow$$ This is in Hermitian conjugate form.

Weyl Spinors :-

We saw that S^{0i} and S^{ij} block diagonal form and it can be seen that the Dirac representation of Lorentz group is reducible.

Can be seen in the Chiral representation

Lets look at the rotation generators

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j]$$

$$= \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

If we take

$$\Omega_{ij} = -\epsilon_{ijk}\phi^k$$

(meaning $\Omega_{12} = -\phi^3$ etc) then

$$\mathcal{D}(\Lambda) = \exp\left(\frac{i}{2}\Omega_{ij}S^{ij}\right)$$

$$= \begin{pmatrix} e^{i\vec{\phi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{+i\vec{\phi}\cdot\vec{\sigma}/2} \end{pmatrix}$$

Consider a rotation by 2π about x^3 axis meaning $\vec{\varphi} = (0, 0, 2\pi)$

and spinor rotation matrix becomes

$$\underbrace{S[\Lambda]}_{\text{rotation.}} = \begin{pmatrix} e^{i\pi\sigma^3} & 0 \\ 0 & e^{i\pi\sigma^3} \end{pmatrix} = -1$$

$\therefore$ Under a 2π rotation

$$\psi^{d}(x) \longmapsto -\psi^{(d)}(x)$$

which usually donot happen to a vector

For Boosts :-

$$S^{0i} = \frac{1}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

Writing boost parameters $\Omega_{i0} = -\Omega_{0i} = \chi_i$ we have

$$\underbrace{D[\Lambda]}_{\text{boost}} = \begin{pmatrix} e^{+\vec{\chi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{-\vec{\chi}\cdot\vec{\sigma}/2} \end{pmatrix}$$

These reducible representation decomposes into two irreducible representation that act on two component spinors $u_\pm$ which is the chiral representation is given by

$$\psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$$

The two component object $u_\pm$ are called Weyl spinors or chiral spinors. They transform in the same way under rotation but oppositely under boosts

$$u_+ \xrightarrow{R} e^{i\vec{\varphi}\cdot\vec{\sigma}/2}\, u_+$$

$$u_- \xrightarrow{B} e^{\pm\vec{\chi}\cdot\vec{\sigma}/}\, u_-$$

Under parity rotations donot change but boost changes a sign and we can see that $P: u_+ \to u_-$

In Group theory language u_+ is in $\left(\frac{1}{2}, 0\right)$ representation of Lorentz group and u_- is in $\left(0, \frac{1}{2}\right)$ representation. The Dirac spinor ψ lies in $\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$ representation. Strictly speaking the spinor is a representation of the double-cover of the Lorentz group $SL(2, \mathbb{C})$

Now as we have seen that the transformation laws of weyl spinors under infinitesimal rotation θ and boost β are

$$u_+ \longrightarrow \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2} - \vec{\chi} \cdot \frac{\vec{\sigma}}{2}\right) u_+$$

$$u_- \longrightarrow \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2} + \vec{\chi} \cdot \frac{\vec{\sigma}}{2}\right) u_-$$

Let us now define

$$\sigma^\mu \equiv (1, \sigma) \quad \text{and} \quad \bar{\sigma}^\mu \equiv (1, -\sigma)$$

$$\sigma^\mu \equiv (1, \sigma)$$

Now we know

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$\therefore \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

$\partial_\mu \bar{\psi} =$

$\not{A} = A_\mu \partial^\mu$

$\bar{\psi} = \psi^\dagger \gamma^0$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

The Dirac lagrangian is

$$\mathcal{L} = \bar{\psi}\,(i \underbrace{\partial_\mu \gamma^\mu}_{\not{\partial}} - m)\,\psi$$

When $\psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$ then $\psi^\dagger = \left(u_+^\dagger \;\; u_-^\dagger\right)$

Now

$$\Rightarrow \quad \bar{\Psi} = \begin{pmatrix} u_+^\dagger & u_-^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} u_-^\dagger & u_+^\dagger \end{pmatrix}$$

$$\mathcal{L} = \begin{pmatrix} u_-^\dagger & u_+^\dagger \end{pmatrix} \left(i \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \partial_\mu - m \right) \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$$

$$= \begin{pmatrix} u_-^\dagger & u_+^\dagger \end{pmatrix} \left(i \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} \partial_\mu u_+ \\ \partial_\mu u_- \end{pmatrix} - \begin{pmatrix} m u_+ \\ m u_- \end{pmatrix} \right)$$

$$= \begin{pmatrix} u_-^\dagger & u_+^\dagger \end{pmatrix} \left[i \begin{pmatrix} \sigma^\mu \partial_\mu u_- \\ \bar{\sigma}^\mu \partial_\mu u_+ \end{pmatrix} - \begin{pmatrix} m u_+ \\ m u_- \end{pmatrix} \right]$$

$$\boxed{\mathcal{L} = i \left(u_-^\dagger \sigma^\mu \partial_\mu u_- + u_+^\dagger \bar{\sigma}^\mu \partial_\mu u_+ \right) - m \left(u_-^\dagger u_+ + u_+^\dagger u_- \right)}$$

The equation of motion for u_+ and u_- are then

$$u_+ : \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial_\mu u_+} \right) = \frac{\partial \mathcal{L}}{\partial u_+}$$

$$\Rightarrow \partial_\mu \left(i u_+^\dagger \bar{\sigma}^\mu \right) = - m u_-^\dagger$$

$$\Rightarrow \boxed{i \bar{\sigma}^\mu \partial_\mu u_+^\dagger + m u_-^\dagger = 0}$$

They are the Weyl Equations

$$u_- : \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial_\mu u_-} \right) = \frac{\partial \mathcal{L}}{\partial u_-}$$

$$\Rightarrow i \partial_\mu \left(u_-^\dagger \sigma^\mu \right) = - m u_+^\dagger$$

$$\Rightarrow \boxed{i \sigma^\mu \partial_\mu u_-^\dagger + m u_+^\dagger = 0}$$

Dirac Matrices and Dirac Field Bilinears

We have seen above that $\bar{\psi}\psi$ is a Lorentz scalar. It is also can be shown that $\bar{\psi}\gamma^\mu\psi$ is a 4-vector. Now lets ask a general question, consider the Expression $\bar{\psi}\Gamma\psi$ where Γ is 4×4 matrix

Can we decompose this expression into terms that have definite transformation properties under the Lorentz group?. The ans is Yes!

If we write Γ in terms of the following basis defined as anti-symmetric combination of γ-matrices

$$\mathbb{1}$$

$$\gamma^\mu$$

$$\gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu,\gamma^\nu] \equiv \gamma^{[\mu}\gamma^{\nu]} = -i\sigma^{\mu\nu}$$

$$\gamma^{\mu\nu\rho} = \gamma^{[\mu}\gamma^\nu\gamma^{\rho]}$$

$$\gamma^{\mu\nu\rho\sigma} = \gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\sigma]}$$

Lets look at the Lorentz transformation properties of these matrices...

Eg:-

$$\bar{\psi}\gamma^{\mu\nu}\psi \longrightarrow \left(\bar{\psi}\Lambda_{\frac{1}{2}}^{-1}\right)\left(\frac{1}{2}[\gamma^\mu,\gamma^\nu]\right)\left(\Lambda_{\frac{1}{2}}\psi\right)$$

$$\boxed{\psi \to \Lambda_{\frac{1}{2}}\psi}$$

$$= \frac{1}{2}\bar{\psi}\Big(\underbrace{\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}}}\underbrace{\Lambda_{\frac{1}{2}}^{-1}\gamma^\nu\Lambda_{\frac{1}{2}}}$$

$$\gamma^0\psi^\dagger \to \psi^\dagger\Lambda_{\frac{1}{2}}^{-1}\gamma^0$$

$$- \underbrace{\Lambda_{\frac{1}{2}}^{-1}\gamma^\nu\Lambda_{\frac{1}{2}}}\underbrace{\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}}}\Big)\psi$$

$$\boxed{\bar{\psi} \Rightarrow \bar{\psi}\Lambda_{\frac{1}{2}}^{-1}}$$

as found before

$$= \frac{1}{2}\bar{\psi}\left(\Lambda^\mu{}_\alpha\gamma^\alpha\Lambda^\nu{}_\beta\gamma^\beta - \Lambda^\nu{}_\beta\gamma^\beta\Lambda^\mu{}_\alpha\gamma^\alpha\right)\psi$$

$$\boxed{\bar{\psi}\gamma^{\mu\nu}\psi \Rightarrow \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\,\bar{\psi}\gamma^{\alpha\beta}\psi}$$

Transformation property of this basis.

let us define a new gamma matrices

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$= \frac{-i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$$

$$= \frac{-i}{4\times3\times2}\Big[\epsilon^{0123}\gamma_0\gamma_1\gamma_2\gamma_3$$

24

$$-\epsilon^{0132}\gamma_0\gamma_1\gamma_3\gamma_2 + \cdots\Big]$$

0123
0132
0312
0321
0213
0231 — 6
1023
1032
1203
1230
1302
1320 — 6

So total possibility $= 6\times4 = 24$

Now,

$$\boxed{\gamma^{\mu\nu\rho\sigma} = -i\,\epsilon^{\mu\nu\rho\sigma}\gamma^5}$$

and

$$\boxed{\gamma^{\mu\nu\rho} = +i\,\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma^5}$$

How??

The properties of γ^5 are—

(i) $(\gamma^5)^\dagger = \gamma^5$

(ii) $(\gamma^5)^2 = \mathbb{1}$

(iii) $\{\gamma^5, \gamma^\mu\} = 0$

commute with all the γ^μ

$$\Rightarrow \boxed{\{\gamma^5, S^{\mu\nu}\} = 0}$$

This statement says that Dirac representation must be reducible

$\{\gamma^\mu \gamma^\nu\} = 2\eta^{\mu\nu}$

We know

$(\gamma^i)^\dagger = -\gamma^i$

$(\gamma^0)^\dagger = \gamma^0$

(Schur's Lemma)

In the present basis

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$ in block diagonal form

Let us now re-write the table and use some standard notation

Thus

$\bar{\psi}\psi$	1	scalar	1
$\bar{\psi}\gamma^\mu\psi$	γ^μ	vector	4
$\bar{\psi} S^{\mu\nu}\psi$	$S^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$	tensor	6
$\bar{\psi}\gamma^5\psi$	$\gamma^\mu\gamma^5$	pseudo-vector / axial vector	4
$\bar{\psi}\gamma^\mu\gamma^5\psi$	γ^5	pseudo-scalar	1
			16

They are spinors bilinear

They are called pseudo coz they transform like a vector and scalar under continuous Lorentz transformation but with an additional sign change under parity transformation, we shall see later

The total no. of bilinears is 16 from which we can form 4-component object.

By using γ^5 we can start to add new terms in the Lagrangian to construct new theories. Typically such terms will break parity invariance of theory but we can make $\phi\bar{\psi}\gamma^5\psi$

where ϕ itself is pseudoscalar

Nature makes use of this parity violating interactions by using γ^5 in the weak force. A theory which treats $\psi_\pm$ on an equal footing is called vector-like theory.

A theory which treats $\psi_\pm$ differently is called chiral theory.

Since $(\gamma^5)^2 = 1$ We can make a Lorentz invariant Projection operator

$$P_\pm = \frac{1}{2}(1 \pm \gamma^5) \quad \text{such that}$$

$$P_+^2 = P_+ \quad \text{and} \quad P_-^2 = P_- \quad \text{and} \quad P_+P_- = 0.$$ Lets look at how P_+ and P_- act on the weyl spinor. We can also now define Chiral Spinors using γ^5 which is

$$\psi_\pm = P_\pm \psi$$ which forms the irreducible representation of the Lorentz group. ψ_+ is called the left handed spinors and ψ_- is called the right handed spinors.

Parity

The spinors $\psi_\pm$ are related to each other by parity. Lets understand this for a bit. lets focus on two discrete symmetries

Time Reversal T : $x^0 \rightarrow -x^0$; $x^i \longrightarrow x^i$

Parity P : $x^0 \rightarrow x^0$; $x^i \rightarrow -x^i$

We shall discuss Parity now and it is an important symmetry which play a big deal in weak interactions. Under parity the left and the right spinors are exchanged. We found earlier that under boosts and rotation u_+ and u_- transforms in a way that under parity they exchange. and in the new notation we shall see that $\psi_\pm = \frac{1}{2}(1 \pm \gamma^5)\psi = u_\mp$

left and right handed spinors and so under Parity

$$\boxed{P : \psi_\pm(x,t) \longmapsto \psi_\mp(-x,t)}$$

Now from vectors and pseudo-vector matrices we can form currents out of Dirac field bilinears:-

We defined it

$$\begin{cases} J^{\mu}(x) = \bar{\psi}(x)\gamma^{\mu}\psi(x) \\ J^{\mu 5}(x) = \bar{\psi}(x)\gamma^{\mu}\gamma^{5}\psi(x) \end{cases}$$

$$\bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0$$

$$-i\partial_{\mu}\bar{\psi}\gamma^{\mu} - m\bar{\psi} = 0$$

$$\Rightarrow i\partial_{\mu}\bar{\psi}\gamma^{\mu} + m\bar{\psi} = 0$$

Let us now compute

$$\partial_{\mu}J^{\mu} = (\partial_{\mu}\bar{\psi})\gamma^{\mu}\psi(x) + \bar{\psi}\gamma^{\mu}\partial_{\mu}\psi$$

If ψ satisfy the Dirac equation then

$$\Rightarrow \gamma^{\mu}\partial_{\mu}\bar{\psi} = +im\bar{\psi}$$

$$= (im\bar{\psi})(\gamma^{\mu})^{-1}\gamma^{\mu}\psi(x)$$

$$+ \bar{\psi}(-im\psi)\gamma^{\mu}(\gamma^{\mu})^{-1}$$

$$= 0$$

Thus J^{μ} is always conserved if $\psi(x)$ satisfy Dirac equation. When we couple the Dirac field to EM field, J^{μ} will become the electric current density. Similarly one can compute

$$\partial_{\mu}J^{\mu 5}(x) = (\partial_{\mu}\bar{\psi}(x))\gamma^{\mu}\gamma^{5}\psi(x)$$

$$+ \bar{\psi}(x)\gamma^{\mu}\gamma^{5}\partial_{\mu}\psi(x)$$

$$= (im\bar{\psi})\cancel{\gamma^{\mu}}\gamma^{5}\psi(x)\cancel{(\gamma^{\mu})^{-1}}$$

$$+ \psi(x)\cancel{\gamma^{\mu}}\gamma^{5}(-im\bar{\psi})\cancel{(\gamma^{\mu})^{-1}}$$

$$= im\bar{\psi}\gamma^{5}\psi - im\,\psi(x)\gamma^{5}\bar{\psi}(x)$$

If $m=0$ then $J^{\mu 5}$ is conserved (called axial vector current)

Majorana Fermions

The spinor ψ is a complex object. Now this is so because the representation that we are dealing with i.e $S(\Lambda)$ is also complex. This means if we want to make ψ real for example by imposing $\psi = \psi^{*}$ then it wouldnot stay that way if we Lorentz transform it. However there is a way to make the Dirac spinor components real. Lets look at the following basis which satisfy Clifford algebra:-

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}$$

The special thing about this basis are that they are purely imaginary

$$(\gamma^\mu)^* = -\gamma^\mu$$

This means the generators $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ and hence $S[\Lambda]$ are real.

So with this basis of the Clifford algebra we can work with the real spinor simply by imposing

$$\psi = \psi^*$$

which is preserved under Lorentz transformation and such spinors are called "Majorana Spinors"

Now suppose we are working / using a general basis for the Clifford algebra which satisfies

$$(\gamma^0)^\dagger = \gamma^0$$

$$(\gamma^i)^\dagger = -\gamma^i$$

We shall now define charge conjugation of Dirac spinors ψ which is

$$\psi^{(c)} = C\psi^*$$

Here C is a 4×4 matrix satisfying

$$C^\dagger C = \mathbb{1} \quad \text{and} \quad C^\dagger \gamma^\mu C = -(\gamma^\mu)^*$$

$$\Rightarrow \boxed{C^\dagger = C^{-1}} \qquad \Rightarrow \gamma^\mu C = -C(\gamma^\mu)^*$$

Let us now check this definition and how it transforms under Lorentz transformation

$$\psi^{(c)} \xmapsto{LT} C[\mathcal{D}(\Lambda)\psi]^*$$

$$= C\psi^*[\mathcal{D}(\Lambda)]^*$$

$$= C\psi^*\left[\exp\left(\frac{i}{2}\Omega_{\mu\nu}[S^{\mu\nu}]\right)\right]^*$$

$$= C\psi^*\left[\exp\left(\frac{i}{2}\Omega_{\mu\nu}\left(\frac{i}{4}[\gamma^\mu\gamma^\nu]\right)\right)\right]^*$$

$$= +\mathcal{D}[\Lambda]C\psi^*$$

$$= \mathcal{D}[\Lambda]\psi^{(c)}$$ Thus $\psi^{(c)}$ transforms nicely under LT. In fact $\psi^{(c)}$ satisfies the Dirac equation also.

$$(i\partial_\mu\gamma^\mu - m)\psi = 0 \Rightarrow (-i\partial_\mu(\gamma^\mu)^* - m)\psi^* = 0$$

$$\Rightarrow C(-i\partial_\mu(\gamma^\mu)^* - m)\psi^* = 0$$

$$\Rightarrow i\partial_\mu\gamma^\mu C\psi^* - mC\psi^* = 0$$

$$\Rightarrow \quad \boxed{(-i\,\partial_\mu \gamma^\mu - m)\,\psi^{(c)} = 0}$$

Fact :- After quantization the Majorana spinor gives rise to a fermions that is its own anti-particle. Also what is this matrix C. Well for a given representation we can find it easily. For Majorana basis where γ are pure imaginary we have

$$C_{Maj} = 1$$

Free Particle Solution of the Dirac Equation

Since Dirac field ψ obeys the Klein-Gordon equation we know that it must be linear combination of plane waves

$$\psi(x) = u(p)\, e^{-ip\cdot x} \qquad \text{where } p^2 = m^2$$

lets consider only on the positive frequency $p^0 > 0$. Now $u(p)$ must satisfy the Dirac Equation

$$p_0^2 - \vec{p}^{\,2} = m^2$$

$$(i\,\partial_\mu \gamma^\mu - m)\psi = 0$$

$$\Rightarrow \quad (i\,\partial_\mu \gamma^\mu - m)\, u(p)\, e^{-ip\cdot x} = 0$$

$$\Rightarrow \quad (i\gamma^\mu(-ip_\mu) - m)\, u(p) = 0$$

$$\Rightarrow \quad (\gamma^\mu p_\mu - m)\, u(p) = 0$$

lets look at this in the rest frame where $p = p_0 = m$ then we have

$$(\gamma^0 p_0 - m)\, u(p) = 0$$

$$\Rightarrow \quad m(\gamma^0 - 1)\, u(p) = 0$$

$$\Rightarrow \quad m\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) u(p) = 0$$

$$\Rightarrow \quad m \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} u(p) = 0$$

Let $u(p) = \begin{pmatrix} a \\ b \end{pmatrix}$ $\therefore \begin{pmatrix} -m & m \\ m & -m \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$

$$\Rightarrow \begin{pmatrix} -ma + mb \\ ma - mb \end{pmatrix} = 0$$

$$\Rightarrow \quad ma = mb \Rightarrow \boxed{a = b}$$

Thus $\boxed{u(p_0) = \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}}$ where ξ is a two component spinor.

This factor we have added for future references.

This is solution in the rest frame,

the general solution can be found by boosting with $\Lambda_{1/2}$:

In the rest frame $p = (m, 0)$. Now lets lorentz transform it

We have $\begin{pmatrix} c\bar{t} \\ \bar{x} \end{pmatrix} = \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix}\begin{pmatrix} ct \\ x \end{pmatrix}$ $\eta =$ rapidity

boost along x-axis

$$\begin{pmatrix} c\bar{t} \\ \bar{x} \end{pmatrix} \sim \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix}\right]\begin{pmatrix} ct \\ x \end{pmatrix}$$

$$\sim \left[1 + \eta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]\begin{pmatrix} ct \\ x \end{pmatrix}$$

Now 4-momentum vector in the infinitesimal form transforms as

$$\begin{pmatrix}E\\ p^3\end{pmatrix} = \left[1+\eta\begin{pmatrix}0&1\\1&0\end{pmatrix}\right]\begin{pmatrix}m\\0\end{pmatrix}$$

where η is infinitesimal parameter. For finite η we have

$$\begin{pmatrix}E\\ p^3\end{pmatrix} = \exp\left[\eta\begin{pmatrix}0&1\\1&0\end{pmatrix}\right]\begin{pmatrix}m\\0\end{pmatrix}$$

$$= \left[\cosh\eta\begin{pmatrix}1&0\\0&1\end{pmatrix} + \sinh\eta\begin{pmatrix}0&1\\1&0\end{pmatrix}\right]\begin{pmatrix}m\\0\end{pmatrix}$$

$$= \begin{pmatrix}m\cosh\eta\\ m\sinh\eta\end{pmatrix}$$

We shall now that

$$\boxed{\begin{aligned} S^{0i} &= -\frac{i}{2}\begin{pmatrix}\sigma^k&0\\0&-\sigma^k\end{pmatrix}\\ \Lambda_{\frac{1}{2}} &= \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\end{aligned}}$$

Now lets apply the same boosts to $u(p)$.

$$u(p) = \exp\left[+\frac{i}{2}\eta\left(+\frac{i}{2}\right)\times 2\begin{bmatrix}\sigma^3&0\\0&-\sigma^3\end{bmatrix}\right]\sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}$$

$$= \exp\left[-\frac{1}{2}\eta\begin{pmatrix}\sigma^3&0\\0&-\sigma^3\end{pmatrix}\right]\sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}$$

$$= \left\{\cosh\left(\frac{1}{2}\eta\right)\begin{pmatrix}1&0\\0&1\end{pmatrix} - \sinh\left(\frac{1}{2}\eta\right)\begin{pmatrix}\sigma^3&0\\0&-\sigma^3\end{pmatrix}\right\}\sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}$$

$$= \left[\begin{pmatrix}\cosh(\eta/2)&0\\0&\cosh(\eta/2)\end{pmatrix} - \begin{pmatrix}\sinh\left(\frac{\eta}{2}\right)\sigma^3&0\\0&-\sinh\left(\frac{\eta}{2}\right)\sigma^3\end{pmatrix}\right]\sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}$$

$$= \begin{pmatrix} \cosh(\eta/2) - \sinh(\eta/2)\sigma^3 & 0 \\ 0 & \cosh\left(\frac{\eta}{2}\right) + \sinh\left(\frac{\eta}{2}\right)\sigma^3 \end{pmatrix} \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$= \begin{pmatrix} \left[\cosh\left(\frac{\eta}{2}\right) - \sinh\left(\frac{\eta}{2}\right)\sigma^3\right] \xi\sqrt{m} \\ \left[\cosh\left(\frac{\eta}{2}\right) + \sinh\left(\frac{\eta}{2}\right)\sigma^3\right] \xi\sqrt{m} \end{pmatrix}$$

$\cosh\eta$
$= \frac{e^{\eta} + e^{-\eta}}{2}$
$\Rightarrow 2\cosh x = e^{x} + e^{-x}$

$\sinh x$
$= \frac{e^{x} - e^{-x}}{2}$
$\Rightarrow 2\sinh x = e^{x} - e^{-x}$

$$= \begin{pmatrix} \left[\frac{e^{\eta/2} + e^{-\eta/2}}{2} - \left(\frac{e^{\eta/2} - e^{-\eta/2}}{2}\right)\sigma^3\right] \xi\sqrt{m} \\ \left(\frac{e^{\eta/2} + e^{-\eta/2}}{2}\right) + \left[\left(\frac{e^{\eta/2} - e^{-\eta/2}}{2}\right)\sigma^3\right] \xi\sqrt{m} \end{pmatrix}$$

$$= \begin{pmatrix} \left[e^{\eta/2}\left(\frac{1-\sigma^3}{2}\right) + e^{-\eta/2}\left(\frac{1+\sigma^3}{2}\right)\right] \xi\sqrt{m} \\ \left[e^{\eta/2}\left(\frac{1+\sigma^3}{2}\right) + e^{-\eta/2}\left(\frac{1-\sigma^3}{2}\right)\right] \xi\sqrt{m} \end{pmatrix}$$

Now
$E = m\cosh\eta$
$p^3 = m\sinh\eta$

$$= \begin{pmatrix} \left[\sqrt{E+p^3}\left(\frac{1-\sigma^3}{2}\right) + \sqrt{E-p^3}\left(\frac{1+\sigma^3}{2}\right)\right] \xi \\ \left[\sqrt{E+p^3}\left(\frac{1+\sigma^3}{2}\right) + \sqrt{E-p^3}\left(\frac{1-\sigma^3}{2}\right)\right] \xi \end{pmatrix}$$

$E + p^3 = \sqrt{m}\sqrt{\cosh\eta + \sinh\eta}$
$= \sqrt{m}\, e^{\eta/2}$

Now lets take the square of first term

$\sigma^\mu = (1, \sigma^i)$
$\bar{\sigma}^\mu = (1, -\sigma^i)$

$$\left(\sqrt{E+p^3}\left(\frac{1-\sigma^3}{2}\right) + \sqrt{E-p^3}\left(\frac{1+\sigma^3}{2}\right)\right)^2$$

$$= (E+p^3)\left(\frac{1-\sigma^3}{2}\right)^2 + (E-p^3)\left(\frac{1+\sigma^3}{2}\right)^2 + 2\sqrt{E+p^3}\sqrt{E-p^3}\left(\frac{1-\sigma^3}{2}\right)\left(\frac{1+\sigma^3}{2}\right)$$

$$= \frac{(E+p^3)}{4}\left(1-2\sigma^3+1\right) + \frac{(E-p^3)}{4}\left(1+2\sigma^3+1\right)$$

$$+ 2\frac{\sqrt{E^2-(p^3)^2}}{4}\cancel{\left(1-(\sigma^3)^2\right)}$$

$$= \frac{E+p^3}{2}\left(1-\sigma^3\right) + \frac{E-p^3}{2}\left(1+\sigma^3\right)$$

$$= \frac{E+p^3}{2} - \frac{E\sigma^3+p^3\sigma^3}{2} + \frac{E-p^3}{2} + \frac{E\sigma^3-p^3\sigma^3}{2}$$

$$= \frac{E}{2}+\frac{E}{2} - \frac{p^3\sigma^3}{2} - \frac{p^3\sigma^3}{2}$$

$$= E - p^3\sigma^3$$

We shall now take the square root of it

$$= \sqrt{E-p^3\sigma^3} = \sqrt{p\cdot\sigma}$$

$\therefore$ The solution can be written as

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\;\xi \\ \sqrt{p\cdot\bar{\sigma}}\;\xi \end{pmatrix}$$

Lets see one identity which will be useful later on —

$$(p\cdot\sigma)(p\cdot\bar{\sigma}) = \left(p_0 - p_i\sigma^i\right)\left(p_0 + p_j\sigma^j\right)$$

$$= p_0^2 - p_i p_j \sigma^i \sigma^j$$

$$= p_0^2 - p_i p_j \delta^{ij}$$

$$= p_0^2 - \bar{p}^2 = m^2$$

We often work with specific spinors ξ. A useful choice is σ^3

For example $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then $\xrightarrow{}$ spin up along 3-axis

for boost along x^3

$$u_p = \begin{pmatrix} \sqrt{p \cdot \sigma}\, \xi \\ \sqrt{p \cdot \bar{\sigma}}\, \xi \end{pmatrix} = \begin{pmatrix} \sqrt{E_0 - p_3 \sigma^3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{E_0 + p_3 \sigma^3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

Now

$$\sqrt{E_0 - p_3 \sigma^3} = \sqrt{E_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - p_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} = \sqrt{\begin{pmatrix} E_0 - p_3 & 0 \\ 0 & E_0 + p_3 \end{pmatrix}}$$

$$= \begin{pmatrix} \sqrt{E_0 - p^3} & 0 \\ 0 & \sqrt{E_0 + p_3} \end{pmatrix}$$

$$\Rightarrow \sqrt{E_0 - p_3 \sigma^3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \sqrt{E_0 - p^3}$$

$$\Rightarrow \sqrt{E_0 + p_3 \sigma^3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \sqrt{E_0 + p^3}$$

$$\therefore \quad u_p = \begin{pmatrix} \sqrt{E_0 - p^3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{E_0 + p^3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

We have $E_0^2 - (p^3)^2 = (m)^2$

$$\Rightarrow (E_0 - p^3)(E_0 + p^3) = m^2$$

$$\Rightarrow \sqrt{E_0 - p^3}\sqrt{E_0 + p^3} = m$$

For massless particle we have

$$E_0^2 - (p^3)^2 = 0$$

$$\Rightarrow E_0^2 = (p^3)^2$$

$$\Rightarrow E_0 = p_3$$

$$= \sqrt{2E} \begin{pmatrix} 0 \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

For $\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow$ spin down along the 3-axis we have

$$u(p) = \begin{pmatrix} \sqrt{E + p^3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \sqrt{E - p^3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \xrightarrow{\text{massless}} \sqrt{2E} \begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \end{pmatrix}$$

Helicity :- The helicity operator is the projection of the angular momentum along the direction of momentum.

$$h = \frac{1}{2} \epsilon_{ijk} \hat{p}^i S^{jk} \longrightarrow \text{rotation generator}$$

$$= \frac{1}{2} \hat{p}_i \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

The massless field $\xi^T = (1, 0)$ has helicity $1/2$ and we say it is right handed. The field $\xi^T = (0, 1)$ has helicity $-1/2$ and is left-handed

The helicity of a massive particle depends on the frame in which its momentum is in the opposite direction (but its spin is unchanged). For a massless particle which travels at speed of light, one cannot perform such a boost.

Inner and Outer Products

Let us introduce a notation ξ^s where $s = 1, 2$ for the two component spinors

$\xi^s \rightarrow$ eigenstates of σ^3

$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. What we have seen now is that for a plane wave with positive frequency, the two independent solution can be written as

$$u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \, \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \, \xi^s \end{pmatrix}$$

Since we knew $\psi^+ \psi$ is not Lorentz invariant so let us check

$$u^+ u = \left(\xi^+ \sqrt{p \cdot \sigma}, \, \xi^+ \sqrt{p \cdot \bar{\sigma}} \right) \cdot \begin{pmatrix} \sqrt{p \cdot \sigma} \, \xi \\ \sqrt{p \cdot \bar{\sigma}} \, \xi \end{pmatrix}$$

$$= \xi^\dagger \sqrt{p\cdot\sigma}\,\sqrt{p\cdot\sigma}\,\xi + \xi^\dagger \sqrt{p\cdot\bar\sigma}\,\sqrt{p\cdot\bar\sigma}\,\xi$$

$$= \xi^\dagger\, p\cdot\sigma\, \xi + \xi^\dagger\, p\cdot\bar\sigma\, \xi$$

$$= \xi^\dagger E_p \xi - \cancel{\xi^\dagger \vec{p}_i \sigma^i \xi} + \xi^\dagger E_p \xi + \cancel{\xi^\dagger p_i \sigma^i \xi}$$

$$= 2\xi^\dagger E_p \xi$$

$$= 2E_p\, \xi^\dagger \xi$$

To make it Lorentz scalar we define

$\bar u(p) = u^\dagger(p)\gamma^0$. Then by calculating

$$\bar u(p) u(p) = \left(\xi^\dagger \sqrt{p\cdot\sigma}\,\gamma^0,\ \xi^\dagger\sqrt{p\cdot\bar\sigma}\,\gamma^0\right)\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi\\ \sqrt{p\cdot\bar\sigma}\,\xi\end{pmatrix}$$

$$= \xi^\dagger\sqrt{p\cdot\sigma}\,\gamma^0\sqrt{p\cdot\sigma}\,\xi + \xi^\dagger\sqrt{p\cdot\bar\sigma}\,\gamma^0\sqrt{p\cdot\bar\sigma}\,\xi$$

Now

$$\sqrt{p\cdot\sigma}\,\gamma^0 \quad \sqrt{p_0 - p_i\sigma^i}\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

Do it for σ^3

$$= \sqrt{p_0 - p_3\begin{pmatrix}1&0\\0&-1\end{pmatrix}}\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$= \sqrt{\begin{pmatrix}p_0-p_3&0\\0&p_0+p_3\end{pmatrix}}\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$= \begin{pmatrix}\sqrt{p_0-p_3}&0\\0&\sqrt{p_0+p_3}\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}0&\sqrt{p_0-p_3}\\ \sqrt{p_0+p_3}&0\end{pmatrix}$$

Now

$$\sqrt{p\cdot\sigma}\,\gamma^0\sqrt{p\cdot\sigma} = \begin{pmatrix}0&\sqrt{p_0-p_3}\\ \sqrt{p_0+p_3}&0\end{pmatrix}\begin{pmatrix}\sqrt{p_0-p_3}&0\\0&\sqrt{p_0+p_3}\end{pmatrix}$$

$$= \begin{pmatrix} 0 & \sqrt{p_0^2 - p_3^2} \\ \sqrt{p_0^2-p_3^2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}$$

$$= m\gamma^0$$

Thus $\boxed{\bar{u}u = 2m\gamma^0 \xi^\dagger \xi}$ → We dont want this so there is some error in the above calculation.

This will be our normalization condition once we also require our two component spinors ξ be normalized as usual $\xi^\dagger \xi = 1$ where we have used the basis ξ^s as orthogonal to each other.

Summary:- The general solution of the Dirac equation can be written as a linear combination of plane waves. The positive frequency of the form

$$\psi(x) = u(p) e^{-ip\cdot x} \quad ; \quad p^2 = m^2 \qquad p^0 > 0$$

There are two linearly independent solution for $u(p)$

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\, \xi^s \\ \sqrt{p\cdot\bar{\sigma}}\, \xi^s \end{pmatrix} \qquad s = 1,2$$

which we normalize a/c to

$$\bar{u}^r(p) u^s(p) = 2m^{\gamma^0} \delta^{rs} \qquad \text{or}$$

$$u^{r\dagger}(p) u^s(p) = 2E_p \delta^{rs}$$

In Exactly the same way we can also find the negative-frequency solution

$$\psi(x) = v(p) e^{+ip\cdot x} \quad , p^2 = m^2 \qquad p^0 > 0$$

notice that we have chosen to put the + sign rather than $p_0 < 0$.

The two linearly independent solutions for $v(p)$ are

$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta^s \end{pmatrix} \qquad s = 1,2$$

↳ another basis of two component spinors.

These solutions are normalized according to

$$\bar{v}^r(p)\,v^s(p) = -2m\,\delta^{rs} \qquad \text{or}$$

$$v^{r\dagger}(p)\,v^s(p) = +2E_p\,\delta^{rs}$$

The u's and v's are also orthogonal to each other

$$\bar{u}^r(p)\,v^s(p) = \bar{v}^r(p)\,u^s(p) = 0$$

But we need to be careful here because

$$u^{r\dagger}(p)\,v^s(p) \neq 0 \quad \text{and} \quad v^{r\dagger}(p)\,u^s(p) \neq 0$$

However it can also be seen that

$$u^{r\dagger}(\vec{p})\,v^s(-\vec{p}) = v^{r\dagger}(-\vec{p})\,u^s(\vec{p}) = 0$$

with the 3-momentum i's $\vec{p}$ taking the opposite sign in the calculations.

Outer Product

Lets compute

$$\sum_{s=1,2} u^s(p)\,\bar{u}^s(p) = \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix}$$

$$\gamma^0 \left(\xi^{s\dagger} \underbrace{\sqrt{p\cdot\sigma}}_{\downarrow} \quad \xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}} \right)$$

$$= \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix} \left(\xi^{s\dagger}\sqrt{p\cdot\bar\sigma} \quad \xi^{s\dagger}\sqrt{p\cdot\sigma} \right)$$

$$= \begin{pmatrix} \sqrt{p\cdot\sigma}\,\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\sigma}\,\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar\sigma}\,\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\bar\sigma}\,\sqrt{p\cdot\sigma} \end{pmatrix}$$

$$= \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar\sigma & m \end{pmatrix}$$

where we have used

$$\sum_{s=1,2} \xi^s \xi^{s\dagger} = 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus we get the desired formula

$$\boxed{\sum_s u^{(s)}(p)\,\bar u^{(s)}(p) = \gamma\cdot p + m}$$

Feynman introduced $\not p \equiv \gamma^\mu p_\mu$ which we shall use later on.

We shall now try to quantize the Dirac field from now-onwards.

— x —