Representation of Lorentz Group and Dirac fields

Sources → Sydney Coleman Notes

→ Peskin

→ Georgie Lie Groups in particle physics.

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Review of Group theory :-

Group theory is the study of symmetries in physics. Symmetries of a physical theory are sets of transformations which leaves some properties of the physical theory invariant. Those transformations can be thought of as elements of group. Given a set of transformations $T_i, T_J, \ldots$ if we perform the first transformation on the physical theory T_i; then perform subsequent transformation T_J; the result from both the transformation can be thought of as a transformation T_K which belong to the same set. We write it as $\underbrace{T_J \cdot T_i}_{\text{first } T_i \text{ then } T_J} = T_K$

Thus we want the Group to follow this closure property.

Def^n :- A Group G is a set with a rule for assigning to every (ordered) pair of elements, a third element obeying

(i) If $f, g \in G$ then $h = fg \in G$

(ii) For $f, g, h \in G$ then $f(gh) = (fg)h$

(iii) $\forall f \in G \ \exists e : ef = fe = f$

(iv) $\forall f \in G$ there exist an inverse f^{-1} such that $ff^{-1} = f^{-1}f = e$

Thus if a group is discrete the group is a multiplication table specifying $g_1 g_2 \ \forall \ g_1 g_2 \in G$

	e	g_1	g_2	g_3	
e	e	g_1	g_2	g_3	…
g_1	g_1	$g_1 g_1$	$g_1 g_2$	$g_1 g_3$	…
g_2	g_2	$g_2 g_1$	$g_2 g_2$	$g_2 g_3$	
⋮	⋮	⋮	⋮		⋱

Def^n :- A Representation of G is a mapping D of the elements of G onto a set of linear operators with the following property :-

(i) $D(e) = \mathbb{1}$ → identity operator in space on which linear operator act

(ii) $\underbrace{D(g_1)\,D(g_2)}_{\text{natural multiplication in the linear space on which linear operators act.}} = D(\underbrace{g_1 g_2}_{\text{group multiplication}})$

Example :- $\mathbb{Z}_3$ [cyclic group of $\underbrace{\text{order}}_{\text{no. of elements in finite group}}$ 3]

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

$\mathbb{Z}_3$ is Abelian since for $g_1 g_2 \in G$

$g_1 g_2 = g_2 g_1$

→ One Representation of $\mathbb{Z}_3$ is [1-dimensional Representation]

$D(e) = 1 \qquad D(a) = e^{2\pi i/3} \qquad D(b) = e^{4\pi i/3}$

→ There is one another way of representing $\mathbb{Z}_3$. The trick is to take group elements and form an orthonormal basis for a vector space $|e\rangle$, $|a\rangle$ and $|b\rangle$. Now we define :-

$$D(g_1)\,|g_2\rangle = |g_1 g_2\rangle$$

This is indeed a representation called the regular representation. lets find the regular representation corresponding to group element e

$$\left.\begin{aligned} |e_1\rangle &= |e\rangle \\ |e_2\rangle &= |a\rangle \\ |e_3\rangle &= |b\rangle \end{aligned}\right\}$$

$$[D(e)]_{ij} = \langle e_i | D(e) | e_j \rangle$$

$$= \langle e_i | \underbrace{e e_j}_{\text{using the definition}} \rangle$$

$$= \langle e_i | e_j \rangle$$

$$= \delta_{ij}$$

Similarly we shall get

$\therefore$

$$D(e) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad D(a) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$D(b) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

<u>Review of Lie Groups</u> :- Suppose the group elements $g \in G$ depends smoothly on a set of continuous parameters — $g(\alpha)$. We shall now parametrize the element in such a way that $\alpha = 0$ corresponds to identity element. If we assume that in some neighbourhood of the identity, the group element is described by function of N real parameters α_a for $a = 1$ to N then

$$g(\alpha)\big|_{\alpha=0} = e \quad \text{and the representation}$$

$$D(\alpha)\big|_{\alpha=0} = \mathbb{1}.$$ In some neighbourhood of Identity Element, we can Taylor Expand $D(\alpha)$

$$D(d\alpha) = 1 + i\, d\alpha_a X_a + \cdots$$ where

$$\boxed{X_a = -i \frac{\partial}{\partial \alpha_a} D(\alpha)\Big|_{\alpha=0}}$$

This are called generators of Group

Now if we go away from the identity in some fixed direction we can just raise the infinitesimal group element

$$D(\alpha) = \lim_{k\to\infty} \Big(1 + i \underbrace{d\alpha_a}_{\frac{\alpha_a}{k}} X_a\Big)^k = e^{i\alpha_a X_a}$$

Thus this mean that we can write group elements in terms of the generators.

However if we multiply two group elements generated by two different linear combination of generators then

$$e^{i\alpha_a X_a} e^{i\beta_b X_b} \neq e^{i(\alpha_a + \beta_a) X_a}$$

But the product in the representation should be some exponential of generator

$$\therefore\ e^{i\alpha_a X_a} e^{i\beta_b X_b} = e^{i\delta_a X_a}$$

We shall now expand both side and equate powers of α and β.

let us check leading order

$$i\delta_a X_a = \ln\big[\underbrace{1 + e^{i\alpha_a X_a} e^{i\beta_b X_b} - 1}_{K}\big] \to (1)$$

$$k = e^{i\alpha_a X_a} e^{i\beta_b X_b} - 1$$

$$= \left(1 + i\alpha_a X_a - \frac{1}{2}(\alpha_a X_a)^2 + \dots\right)\left(1 + i\beta_b X_b - \frac{1}{2}(\beta_b X_b)^2 + \dots\right) - 1$$

$$= i\alpha_a X_a + i\beta_b X_b - \frac{1}{2}(\alpha_a X_a)^2 - \frac{1}{2}(\beta_b X_b)^2 - \alpha_a X_a \beta_b X_b$$

Now

$$i\delta_a X_a = k - \frac{1}{2}k^2$$

$$= i\alpha_a X_a + i\beta_b X_b - \frac{1}{2}(\alpha_a X_a)^2 - \frac{1}{2}(\beta_b X_b)^2 - \alpha_a X_a \beta_b X_b + \frac{1}{2}(\alpha_a X_a + \beta_b X_b)^2$$

$$= i\alpha_a X_a + i\beta_a X_a - \frac{1}{2}[\alpha_a X_a, \beta_b X_b]$$

$$\Rightarrow [\alpha_a X_a \ \beta_b X_b] = \overbrace{-2i(\delta_c - \alpha_c - \beta_c)X_c + \underbrace{\dots}_{\substack{\text{represent}\\ \text{terms that have}\\ \text{more than two}\\ \text{factors of } \alpha \text{ or } \beta}}}^{\text{The whole thing is } i\gamma_c X_c}$$

Let say $a = \{1, 2\}$ then

$$[\alpha_1 X_1 + \alpha_2 X_2, \beta_1 X_1 + \beta_2 X_2]$$

$$= [\alpha_1 X_1, \beta_1 X_1] + [\alpha_1 X_1 + \beta_2 X_2] + [\alpha_2 X_2, \beta_1 X_1] + [\alpha_2 X_2, \beta_2 X_2]$$

$$= \alpha_1\beta_1 \cancel{[X_1, X_1]} + \alpha_1\beta_2 [X_1, X_2] + \alpha_2\beta_1 [X_2 X_1] + \alpha_2\beta_2 \cancel{[X_2, X_2]}$$

$$= \alpha_1\beta_2 [X_1, X_2] + \alpha_2\beta_1 [X_2 X_1]$$

which can be generalized as $\alpha_a \beta_b [X_a, X_b]$

The right hand side can be defined as $i \gamma_c X_c =$

where $\gamma_c = -2(\delta_c - \alpha_c - \beta_c)$

Thus we can define some constants f_{abc} for which

$$\boxed{[X_a, X_b] = i f_{abc} X_c}$$

Generators form an algebra under commutation

Exchanging a and b we get

$$[X_b, X_a] = i f_{bac} X_c$$

$$\Rightarrow -[X_a, X_b] = i f_{bac} X_c$$

$$\Rightarrow \boxed{f_{abc} = -f_{bac}}$$

these are called structure constant

su(2) algebra is familiar

$$[J_j, J_k] = i \epsilon_{jkl} J_l$$

Lets now move towards $SO(3,1)$ group which is the group of orthogonal transformations with determinant 1 which preserves the square of the Minkowski norm

$$x_0^2 - x_1^2 - x_2^2 - x_3^2.$$

let us now look at the transformation of fields under Lorentz group. For a scalar field, the transformation law is given as

$$\phi \longrightarrow \phi(\Lambda^{-1} x)$$

For a vector field A_μ, the transformation law is

$$A_\mu \rightarrow \Lambda_\mu{}^\nu A_\nu(\Lambda^{-1} x)$$

The above fields describe elements with integer spins. But if we want to describe transformation law of half-integer spins we can write a general law for fields

$$\boxed{\phi_a(x) \longrightarrow \underbrace{D_{ab}(\Lambda)}\, \phi_b(\Lambda^{-1}x)}$$

Representation of Lorentz transformation.

This can be more complicated matrix depending on what sort fields we are describing. For most of our theory, we shall be needing the fields that describes spin ½ particles which are electrons, protons etc..

The Elements of the Lorentz group Λ has certain properties which needs to be satisfied by the representation

① If $\Lambda_1, \Lambda_2 \in \Lambda$ then

$\Lambda_1 \Lambda_2 = \Lambda_3 \in \Lambda$, then we have

$$D(\Lambda_1)\, D(\Lambda_2) = D(\Lambda_1 \Lambda_2) = D(\Lambda_3)$$

Now for Λ and Λ^{-1} we have.

$$D(\Lambda)\, D(\Lambda^{-1}) = D(\Lambda \Lambda^{-1}) = \underbrace{D(\mathbb{I}) = \mathbb{I}}$$

first property of the representation

$$\Rightarrow \boxed{D(\Lambda^{-1}) = [D(\Lambda)]^{-1}}$$

Thus the representation D forms a finite dimensional representation of Lorentz group.

Let us suppose $D(\Lambda)$ is a representation then

$D(\Lambda)' = T\, D(\Lambda)\, T^{-1}$ for any fixed T is also a representation to prove this

$$D(\Lambda_1)' D(\Lambda_2)' = T D(\Lambda_1) T^{-1} T D(\Lambda_2) T^{-1}$$

If two representations are related in this way then we say

$$= T D(\Lambda_1) D(\Lambda_2) T^{-1}$$

$$= T D(\Lambda_1 \Lambda_2) T^{-1}$$

$D(\Lambda) \sim D'(\Lambda)$ (equivalent)

$$= D'(\Lambda_1 \Lambda_2)$$

→ which satisfies multiplicative law of representation.

Thus we can see that given a representation we can always perform similarity transformation to get a different representation which are equivalent to previous one. There is one more way of generating representation from the old one. Suppose $D^{1}(\Lambda)$ and $D^{2}(\Lambda)$ of dim n_1 and dim n_2 are two representations, then we can make.

$$D(\Lambda) = \begin{bmatrix} D^{(1)}(\Lambda) & \\ & D^{(2)}(\Lambda) \end{bmatrix} \equiv D^{(1)}(\Lambda) \oplus D^{(2)}(\Lambda)$$

This too is a representation

Direct Sum

with $\dim D(\Lambda) = \dim D^{(1)}(\Lambda) + \dim^{(2)}(\Lambda)$

But we are not interested in representation that can be written (reduced) into direct sum. We call such representation "reducible". So our task will be to find all the irreducible finite dimensional representation of the Lorentz Group. Lets first compute the irreducible representation of a subgroup which is Rotation group $SO(3)$.

group of rotation in space R^3 about some axis by some angle.

The rotation matrix R can be labelled by an axis $\vec{n}$ and some angle θ

$$R \in SO(3) : R(\vec{n}\,\theta) = R \qquad 0 \leq \theta \leq \pi$$

We observe that

$$R(\vec{n}\,\theta)\, R(\hat{n}\,\theta') = R(\vec{n}(\theta+\theta'))$$. Thus the representations will also satisfy

$$D(R(\vec{n}\,\theta))\, D(R(\vec{n}\,\theta')) = D(R(\vec{n}(\theta+\theta')))$$

Take the derivative at $\theta'=0$

$$D(R(\vec{n}\,\theta)) \frac{\partial}{\partial \theta'} D(R(\vec{n}\,\theta'))\Big|_{\theta'=0} = \frac{\partial}{\partial(\theta+\theta')} D(R(\vec{n}(\theta+\theta')))\Big|_{\theta'=0}$$

We shall define

$$\frac{\partial}{\partial \theta} D(R(\vec{n}\,\theta))\Big|_{\theta=0} = -i\vec{n}\cdot\mathbb{L}$$

$$-i\vec{n}\cdot\mathbb{L}\, D(R(\vec{n}\,\theta)) = \frac{\partial}{\partial\theta} D(R(\vec{n}\,\theta))$$

$$\Rightarrow \boxed{D(R(\vec{n}\,\theta)) = e^{-i\vec{n}\cdot\vec{\mathbb{L}}\,\theta}}$$ By putting $D(R(\theta=0)) = 1$

generators of the Lorentz group $\{L^i\}$ where $i=1,2,3$

<u>Working out the algebra of the matrices $\{L^i\}$</u>

The transformation of vector $\vec{v}$ about any axis $\vec{n}$ by an infinitesimal rotation by θ is given by

$$v \longrightarrow v + \theta\,\vec{n}\times\vec{v} + \mathcal{O}(v^2)$$

Now the generators $\{L^i\}$ of the group act as an operator in the linear space of the group elements. We shall look at how generatoors transforms under rotation. lets look for a general operator A. It acts on state $|\psi\rangle$ as

$$A|\psi\rangle = |\phi\rangle$$

$$\Rightarrow A\, D(R)^{\dagger} D(R)\, |\psi\rangle = |\phi\rangle$$

$$\Rightarrow \underbrace{D(R)\, A\, D(R)^{\dagger}}_{A'} \underbrace{D(R)\, |\psi\rangle}_{|\psi'\rangle} = D(R)\, |\phi\rangle$$

$$A' \quad |\psi'\rangle = |\phi'\rangle$$

$$\therefore \boxed{A' = D(R)\, A\, D(R)^{\dagger}}$$

any operator transforms like this

Thus for infinitesimal transformation we have

$$(1 - i\hat{n}\theta\cdot\mathbb{L})\, A\, (1 + i\hat{n}\theta\cdot\mathbb{L}) = A'$$

$$\Rightarrow A + A i\hat{n}\theta\cdot L - i\hat{n}\theta\cdot\mathbb{L} A = A'$$

$$\Rightarrow A' = A + i\theta n_k [L_k, A]$$

Now when A is rotationally invariant, then $A' = A$

$$\Rightarrow \boxed{[L_k, A] = 0}$$

For a velocity vector we have

$$\vec{v}' = \vec{v} + i\theta n_k [L_k, \vec{v}]$$

$$\Rightarrow \cancel{v_i} + \varepsilon_{ijk}\theta n_j v_k = \cancel{v_i} + i\theta n_k [L_k, v_i]$$

$$\Rightarrow [L_k, v_i] = -i\varepsilon_{ikj} v_j$$

$$\Rightarrow \boxed{[L_i, v_j] = i\varepsilon_{ijk} v_k}$$

Since $\{\mathbb{L}\}$ also form a vector in the linearspace we shall have the algebra $\boxed{[L_i, L_j] = i\epsilon_{ijk} L_k}$ } The famous angular momentum commutator.

Finding this generators will get us the representation. Thus if we can find up to equivalence and direct sum, all matrices that obey these commutation relations we shall have all the rep of the Rotation group.

Finite Dimensional inequivalent irrep of the Lie algebra of Rotation group are notated by $D^{(s)}(R)$ labelled by an index "s".

$$D^{(s)}(R) = e^{-i\,\vec{n}\,\theta \cdot \underbrace{\mathbb{L}^{(s)}}}$$

triplet of matrices appropriate to spin s.

We have

$$\mathbb{L}^{(s)} \qquad s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots$$

$$\vec{L}^{(0)} = \vec{0}$$

$$\vec{\mathbb{L}}^{(1/2)} = \frac{\vec{\sigma}}{2} \quad \text{where} \quad \sigma = \text{Pauli matrices.}$$

→ The dimension of the representation $D^{(s)}(R)$ is $2s+1$

→ The square of $\mathbb{L}^{(s)}$ is multiple of identity

$$\mathbb{L}^{(s)} \cdot \mathbb{L}^{(s)} = s(s+1)\mathbb{I}$$

If we choose one component of $\mathbb{L}^{(s)}$ lets say $\mathbb{L}_z^{(s)}$ then we shall have $\mathbb{L}_z^{(s)}|m\rangle = m|m\rangle$ where

$$m = -s, -s+1, -s+2, \ldots, s-2, s-1, s$$

Some facts :- ① The representation of lie algebra just listed above not only generates the representation of Rotation group they generate representation upto a phase. The integer s are representation. the half integer s are reps upto a phase. ie they are double valued

$$D^{(s)}\left(R(2\pi \vec{n})\right) = (-1)^{2s} \mathbb{1}$$

② If $D^{(s)}(R)$ is a rep of $SO(3)$ then so is $D^{(s)}(R)^*$.

$$\therefore \quad D^{(s)}(R) \sim D^{(s)}(R)^*$$

③ If we have some sets of fields that transforms under rotation as a irrep $D^{(s_1)}(R)$ and second sets of fields that transforms as another irrep $D^{(s_2)}(R)$ then we can get a new representation given by

$$D^{(s_1)}(R) \otimes D^{(s_2)}(R)$$

But its not necessary a irreducible representation

The dim of the direct product is

$$(2s_1+1)(2s_2+1)$$

There is a rule of how we can break it up into irreducible representations. It is equivalent to direct sum which can be indicated as

$$D^{(s_1)}(R) \otimes D^{(s_2)}(R) \sim \bigoplus \sum_{s=|s_1-s_2|}^{s_1+s_2} D^{(s)}(R)$$

For eg

$$D^{(1/2)}(R) \otimes D^{(1/2)}(R) \sim D^{(0)} \oplus D^{(1)}$$

$\underbrace{\hspace{6em}}$ $\underbrace{\hspace{4em}}$

The product of spinors give two object — a scalar and a vector.

Lorentz Group

Lorentz transformation can be decomposed into a rotation and a boost. A boost $A(\hat{a}\phi)$ along a given axis $\vec{a}$ and rapidity ϕ is a pure Lorentz transformation that takes a particle at rest and changes its velocity to some new value along that axis.

As with rotations, we have

$$A(\hat{a}\,\phi)\,A(\hat{a}\,\phi') = A(\hat{a}\,(\phi+\phi'))$$

By defining

$$-i\,\hat{a}\,\phi\cdot \underset{\sim}{M} = \left.\frac{\partial D(A(\hat{a}\,\phi))}{\partial \phi}\right|_{\phi=0}$$

M is the generator of boosts.

and we shall find that

$$D(A(\hat{a}\,\phi)) = e^{-i\,\hat{a}\cdot M\phi}$$

Thus if we know L and M we know the representation matrix for arbitrary rotation and arbitrary boosts and by multiplication we can find the representation matrix for any general Lorentz transformation. Let us now write all the commutators of L and M. For rotations we have.

$$[L_i, L_j] = i\,\epsilon_{ijk}\,L_k$$

Next

$$[L_i, M_j] = i\,\epsilon_{ijk}\,M_k$$

This tells us that M transforms like a vector.

Now $[M_i, M_j] = -i\,\epsilon_{ijk} L_k$

The minus sign here is very important.

Now to find all the irreducible representation of Lorentz algebra,

We shall now use a trick:- We shall now define

$$J^{\pm} = \frac{1}{2}(L \pm iM) \quad \text{so we have}$$

$$L = J^{+} + J^{-}$$

$$M = -i\,(J^{+} - J^{-})$$

Let us compute the commutation of these new operators and we shall see

$$[J_i^{(-)}, J_j^{(-)}] = i\,\epsilon_{ijk} J_k^{(-)}$$

$$[J_i^{(+)}, J_j^{(+)}] = i\,\epsilon_{ijk} J_k^{(+)}$$

$$[J_i^{(+)}, J_j^{(-)}] = 0$$

Thus $\{J_i^{(+)}\}$ and $\{J_j^{(-)}\}$ commute with each other

The two $\{J_i^{(+)}\}$ and $\{J_j^{(-)}\}$ forms two commuting independent SU(2) algebras. Thus a complete set of irreducible representation of Lorentz group are characterised by two spin quantum no $s+$ and $s-$ one for each J^{+} and J^{-} and written as

$$D^{(s+,\, s-)}(\Lambda) \qquad s_{\pm} = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots$$

The square of these operators J^+ and J^- are multiples of identity

$$J^+ \cdot J^+ = s^+ (s^+ + 1) J^+$$

$$J^- \cdot J^- = s^- (s^- + 1) J^-$$

The complete set of basis is defined by two numbers m_+ and m_- which are the Eigenvalue of J_z^+ and J_z^- respectively such that

$$J_z^{\pm} |m_+ m_-\rangle = m_{\pm} |m_+ m_-\rangle$$

These states are simultaneous eigenstates of commuting operators.

We can always choose our basis such that J^+ and J^- are hermitian matrices and so we can see that L is hermitian but not M $\therefore$ $D(R)$ are unitary but $D(A)$ are not.

Properties of $SO(3,1)$ representation $D^{(s+,s-)}(\Lambda)$

$$\left[D^{(s+,s-)}(\Lambda)\right]^* \sim D^{(s-,s+)}(\Lambda)$$

$$P : D^{(s+,s-)}(\Lambda) \to D^{(s-,s+)}(\Lambda)$$

Parity

This is coz Parity turns L into L and M into $-M$. The operation $M \to -M$ can be thought of as exchanging $J^{(+)}$ and $J^{(-)}$

$$D^{(s+,s-)}(R) \sim \oplus \sum_{s=|s^+-s^-|}^{s^++s^-} D^{(s)}(R)$$

Extras :- Finding the general commutations of the generators of Lorentz group!

We know in quantum mechanics $\xrightarrow{}$ generators of rotation

$$J = x \times p = x \times (-i\nabla)$$

lets write the operators as an antisymmetric tensor

$$J^{ij} = -i(x^i \nabla^j - x^j \nabla^i)$$

where $J^3 = J^{12}$ and so on.. The generalization to 4-vector is

$$\boxed{J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu)}$$

We shall be able to compute now that (A)

$$\boxed{[J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho} + \eta^{\mu\sigma} J^{\nu\rho})}$$

lorentz

Any matrices that are to represent this algebra can be used to find the representation of lorentz group.

Lets find the representation corresponding to spin 1/2. lets use a trick used by Dirac! If we have $n \times n$ matrices γ^μ satisfying the anti-commutation relation which is

Dirac matrices $\leftarrow$ $\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \times \mathbb{1}_{n\times n}}$ (Dirac algebra)

then we could write down an n-dimensional representation of Lorentz algebra. which is

$$\boxed{S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]}$$

We can actually show that this $S^{\mu\nu}$ satisfy (A). This trick can be used for any dimensionality whether Lorentz or Euclidean metric. Lets work it in 3-dimensional Euclidean space where we choose

$$\gamma^j \equiv i\sigma^j \quad \text{(Pauli sigma matrices)}$$

Thus

$$\{\gamma^i, \gamma^j\} = \{i\sigma^i, i\sigma^j\} = -\{\sigma^i, \sigma^j\}$$

$$= \underbrace{-}_{\text{This minus sign is conventional.}} 2\delta_{ij}\mathbb{I}$$

The representation will then be

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j]$$

$$= \frac{i}{4}(-1)[\sigma^i, \sigma^j]$$

$$= \frac{-i}{4} 2i\epsilon^{ijk}\sigma^k$$

$$\underbrace{S^{ij} = \frac{1}{2}\epsilon^{ijk}\sigma^k}_{\text{This is the 2-dimensional representation of rotation group.}}$$

where we have used

$$[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$$

$$[\sigma^i, \sigma^j] = 2\delta^{ij}\mathbb{I}$$

\# We shall need to find Dirac matrices for Minkowski space

One representations in 2×2 block form is

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

This representation is called Weyl or Chiral representation

Thus using

$$S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right]$$

Thus

$$S^{0i} = \frac{i}{4}\left[\gamma^0, \gamma^i\right]$$

$$= \frac{i}{4}\left\{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} - \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$$

$$= \frac{i}{4}\left\{\begin{bmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{bmatrix} - \begin{bmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{bmatrix}\right\}$$

$$\boxed{S^{0i} = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}}$$

This is not hermitian and thus transformation of boosts is not unitary.

Also we can calculate and shall find that

$$S^{ij} = \frac{i}{4}\left[\gamma^i, \gamma^j\right]$$

$$= \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

$$\boxed{S^{ij} \equiv \frac{1}{2}\epsilon^{ijk}\Sigma^k}$$

These are the generators of the Lorentz group and the four component field ψ that transforms under boost and rotation a/c to these generators are called Dirac Spinors.