1st Tutorial [Symmetries]

Noether theorem (Symmetry) :- Lets look at a continuous transformations on the field ϕ, which in infinitesimal form can be written as

$$\phi(x) \longrightarrow \phi'(x) = \phi(x) + \underbrace{\varepsilon}_{\text{infinitesimal parameter}} \Delta\phi(x) \longrightarrow (1)$$

$\Delta\phi(x)$ = some deformation of the field configuration.

Now we shall call this transformation a symmetry if it leaves the EOM invariant. This is insured if the action is invariant under (1)

Generally, we can allow the action to change by a surface term coz the presence of such terms will not affect our derivation of EOM. The Lagrangian must therefore be invariant under (1) upto 4-divergence

$$\delta\mathcal{L} = \partial_\mu F^\mu \qquad \text{for some } F^\mu.$$

This variation means- $\delta\mathcal{L} = \dfrac{\mathcal{L}(\phi') - \mathcal{L}(\phi)}{\varepsilon}$

The variation of Lagrangian is

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)$$

$$\Rightarrow \quad \partial_\mu F^\mu = \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \phi)} \delta\phi \right)$$

$$\Rightarrow \quad \partial_\mu \underbrace{\left(\frac{\partial L}{\partial(\partial_\mu \phi)} \delta\phi - F^\mu \right)}_{J^\mu} = 0$$

current

$$\Rightarrow \quad \boxed{\partial_\mu J^\mu = 0}$$

→ Thus for each continuous symmetry of the Lagrangian we have a conservation la

→ This is the Continuity Equation.

The Charge is

$$Q \equiv \int J^0 \, d^3x \quad \text{constant in time}$$

→ In class Suvrat talked about Lorentz transformation and some internal continuous symmetries of the Lagrangian. We shall discuss about discrete symmetries.

But first lets talk about

Lorentz transformation properties of the charges :-

Now given a conserved current J^μ → I will assume it transforms like a vector field. That is under Lorentz transformation Λ :

$$J^\mu(x) \longmapsto J^{\mu\prime}(x') = \Lambda^\mu{}_\nu J^\nu(\Lambda^{-1} x)$$

We have defined

$$Q \equiv \int d^3x \, J^0(x, 0)$$ [We can define Q at any time since it is time independent so I choose $t=0$]

Because we know how J^μ transforms we know how Q transforms. But is $Q = Q'$? Lets rewrite the charge

$$Q = \int \underbrace{d^4x}_{d} \, \delta(n \cdot x) \underbrace{n \cdot J(x)}$$ This is a fancy way of writing J^0

turning space integral into 4d integral

and $n_\mu = (1, 0, 0, 0)$ is unit vector pointing in time direction so that $n \cdot x = n_\mu x^\mu = x^0$

We can also write this as

$$Q = \int d^4x \, \partial_\mu \theta(n \cdot x) \, J^\mu(x)$$

cot space derivative of theta function is zero. and so

$$\partial_\mu \theta(n \cdot x) = n_\mu \delta(n \cdot x)$$

Lets write the transformed Q as

$$Q' = \int d^4x' \, \delta(n\cdot x) \, n\cdot \underbrace{\Lambda J(\Lambda^{-1}x)}$$

→ active view of transformation.
ie we transform the fields
and then write the new charge.

We donot change the integration
surface at the same time

We define

$x = \Lambda x'$ $\quad$ $n = \Lambda n'$ $\quad$ and so by Lorentz invariance

$$n\cdot x = \Lambda x' \cdot \Lambda n' = n'\cdot x' \quad \text{and}$$

$$n\cdot \Lambda J = \Lambda n' \cdot \Lambda J = n'\cdot J$$

Thus

$$Q' = \int d^4x' \, \delta(n'\cdot x') \, n'\cdot J(x')$$

invariance of d^4x under
Lorentz transformations

$$= \int d^4x \, \delta(n'\cdot x) \, n'\cdot J(x)$$

$$= \int d^4x \, \partial_\mu \Theta(n'\cdot x) J^\mu(x)$$

Now

$$\alpha - \alpha' = \int d^4x \left(\partial_\mu \left[\Theta(n\cdot x) - \Theta(n'\cdot x) \right] \right) J^\mu(x)$$

x^0 $\quad$ $x^{0\prime}$

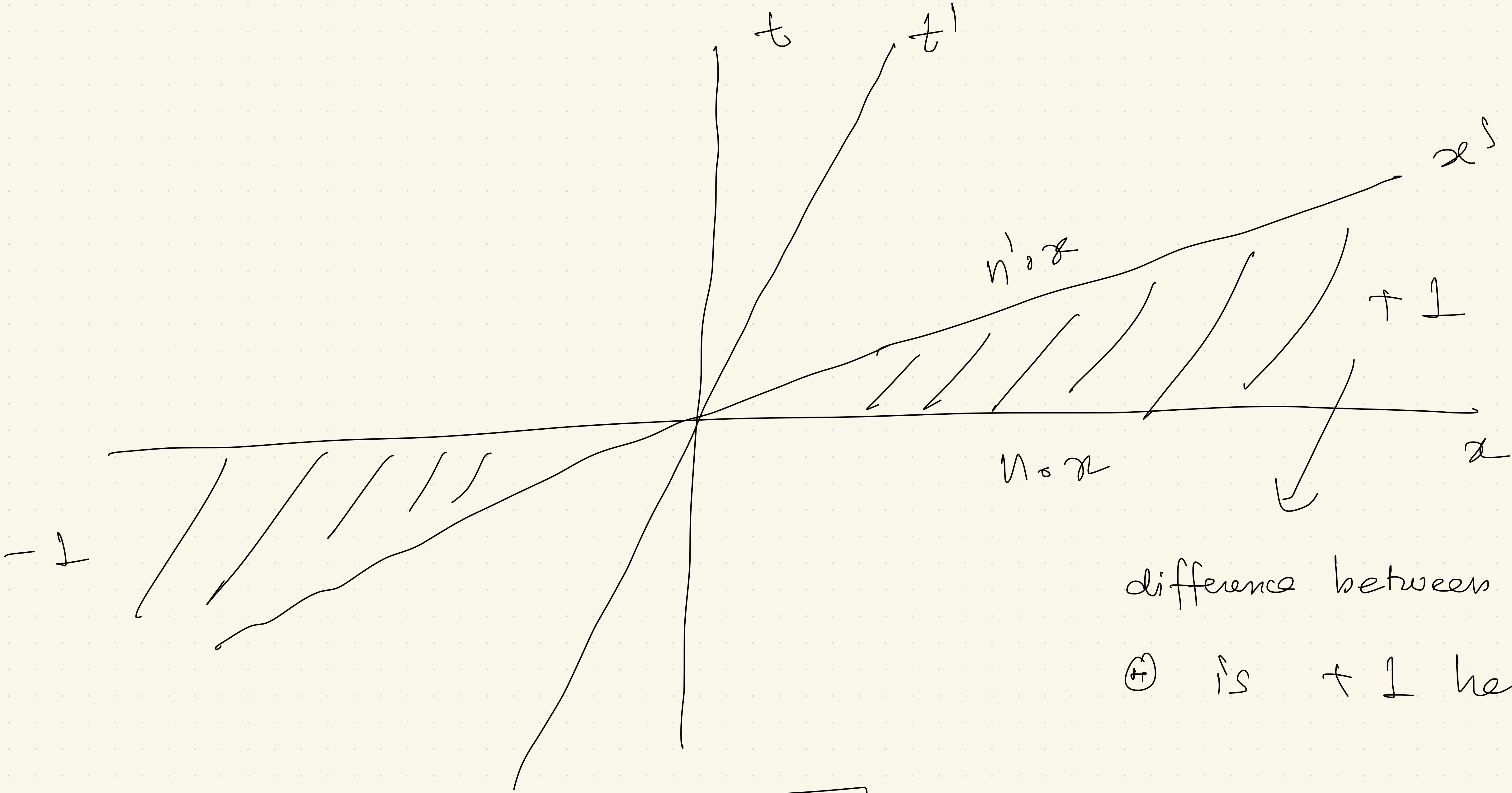
$$= \int d^4x \, \partial_\mu \left\{ \left[\Theta(n\cdot x) - \Theta(n'\cdot x) \right] J^\mu(x) \right\}$$

$$- \int d^4x \left[\Theta(n\cdot x) - \Theta(n'\cdot x) \right] \partial_\mu J^\mu(x)$$

This is the Boundary / surface term. At any fixed x as $t \to \infty$ the quantity becomes zero as both $n\cdot x$ becomes positive and $n'\cdot x$ also becomes positive for each Θ function is 1. Likewise as $t \to -\infty$ both Θ becomes -1 and so is zero.

$$\int dS_\mu \left[\Theta(n\cdot x) - \Theta(n'\cdot x) \right] J^\mu(x)$$

This is zero since $\partial_\mu J^\mu = 0$



difference between two

Θ is $+1$ here.

Thus $\boxed{Q = Q'}$ Q.E.D

Discrete-Symmetry

A discrete symmetry is a transformation where

$\phi(x) \longrightarrow \phi'(x)$ but no parameter in the transformation.

It simply doesnot appear.

For eg:- there is no such thing as parity transformation by 7° there is only parity. It's not like rotation. But still there are these symmetries coz they leave the action invariant.

Charge Conjugation:-

Lets say

$$\mathcal{L} = \frac{1}{2}\left(\partial^\mu \phi^a \partial_\mu \phi^a - \mu^2 \phi^a \phi^a\right)$$

we have two free fields $a = \{1,2\}$ of the same mass.

In the class we said that this system was $SO(2)$ invariant

Rotation group of Euclidean geometry

$$\phi^1 \longrightarrow \phi^1 \cos\lambda + \phi^2 \sin\lambda = \phi^1 + \lambda\phi^2$$

$$\phi^2 \longrightarrow \phi^2 \cos\lambda - \phi^1 \sin\lambda = \phi^2 - \lambda\phi^1$$

$$\Rightarrow \quad \delta\phi^1 = \phi^2$$

$$\Rightarrow \quad \delta\phi^2 = -\phi^1$$

Thus the current is

$$J^\mu = \frac{\partial L}{\partial(\partial_\mu \phi^a)} \delta\phi^a - F^\mu \qquad \text{Here } F^\mu = 0$$

$$J^\mu = (\partial^\mu \phi^1)\phi^2 - (\partial^\mu \phi^2)\phi^1$$

Thus the charge here is

$$\boxed{Q = \int J^0 d^3x = \int (\partial^0 \phi^1)\phi^2 - (\partial^0 \phi^2)\phi^1}$$

$a_p^{(1)}, a_p^{+(2)}$ } type 1 operator

Now we have

$$\phi^a(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p^{(a)} e^{-ip\cdot x} + a_p^{(a)+} e^{ip\cdot x} \right)$$

and

$$[a_p^{(a)}, a_{p'}^{(b)}] = [a_p^{(a)+}, a_{p'}^{(b)+}] = 0$$

$$[a_p^{(a)}, a_{p'}^{(b)+}] = \delta^{ab} \delta^{(3)}(p-p')$$

$\Rightarrow$ Thus we shall get

$$\boxed{Q = i \int \frac{d^3p}{(2\pi)^3} \left[\underbrace{a_p^{(1)\dagger} a_p^{(2)}} - \underbrace{a_p^{(2)\dagger} a_p^{(1)}} \right]}$$

This expression looks nice and it has all the property you would expect for internal symmetry.

this term replaces type 2 particle with type (1)

This term annihilate type 1 particle with type 2

→ Q commutes with Hamiltonian and momentum and also

this Q annihilates the vaccum $\quad Q|0\rangle = 0 \quad$ But Q is not diagonal with

operators $\quad \{a_p^{(a)}\} \quad \{a_p^{(b)\dagger}\}$

$$[Q, a_p^{(a)}] = -i\epsilon^{ab} a_p^{(b)}$$

$$[Q, a_p^{(a)\dagger}] = -i\epsilon^{ab} a_p^{(b)\dagger}$$

Now there is a way to make this commutators look nice.

Lets re-define the annihilation and creation operators which are linear combination of our original $a_p^{(a)}$ and $a_p^{(b)\dagger}$

Def^n :=

Likewise,

$$b_p = \frac{1}{\sqrt{2}} \left(a_p^{(1)} + i a_p^{(2)} \right) \qquad c_p = \frac{1}{\sqrt{2}} \left(a_p^{(1)} - i a_p^{(2)} \right)$$

$$b_p^{\dagger} = \frac{1}{\sqrt{2}} \left(a_p^{\dagger (1)} - i a_p^{\dagger (2)} \right) \qquad c_p^{\dagger} = \frac{1}{\sqrt{2}} \left(a_p^{\dagger (1)} + i a_p^{\dagger (2)} \right)$$

Lets check commutations among these operators.

$$[b_p, c_{p'}^{\dagger}] = \frac{1}{2} \left[a_p^{(1)} + i a_p^{(2)}, \; a_{p'}^{(1)} - i a_{p'}^{(2)} \right]$$

$$= \frac{1}{2} \left[a_p^{(1)}, a_{p'}^{(1)} \right] + \frac{1}{2} \cancel{\left[a_p^{(1)}, - i a_{p'}^{(2)} \right]}$$

$$+ \frac{1}{2} \cancel{\left[i a_p^{(2)}, a_{p'}^{(1)} \right]} - \frac{1}{2} \left[a_p^{(2)}, a_{p'}^{(2)} \right]$$

$$= \frac{1}{2} \delta^{(3)} (p - p') - \frac{1}{2} \delta^{(3)} (p - p') = 0$$

Thus we shall see

$$[b_p^{\dagger}, c_{p'}] = 0$$

Now in this new basis the algebra of Q with b's and c's works out really well

By substitution we shall get

$$Q = \int d^3p \left[b_p^\dagger b_p - c_p^\dagger c_p \right] = N_b - N_c$$

This is similar to Electric charge

The total charge is found by counting the difference between numbers of two particles.

where N_b and N_c are no. of b-type and c-type particles in a given state.

Here

$$[Q, b_p] = -b_p \qquad [Q, b_p^\dagger] = b_p^\dagger \quad (2)$$

$$[Q, c_p] = c_p \qquad [Q, c_p^\dagger] = -c_p^\dagger$$

Thus we have diagonalized Q by writing b and c

Lets look at (2)

$$Q b_p^\dagger - b_p^\dagger Q = b_p^\dagger$$

when acting on ground state $|0\rangle$

$$Q b_p^\dagger |0\rangle = b_p^\dagger |0\rangle$$

$$\Rightarrow \quad Q(b_p^\dagger |0\rangle) = 1(b_p^\dagger |0\rangle)$$ and so for b-type particle the charge $Q = 1$

Till now what we have done is that we have defined new pairs of annihilation and creation operators and wrote the charge and Hamiltonian in terms of that given the fields $\phi^{(1)}$ and $\phi^{(2)}$. But interestingly the same thing can also be achieved by re-defining the fields. Lets define a new field ψ, complex and its adjoint ψ^*

$$\psi = \frac{1}{\sqrt{2}}\left(\phi^1 + i\phi^2\right)$$

$$\psi^* = \frac{1}{\sqrt{2}}\left(\phi^1 - i\phi^2\right)$$

In terms of new creation and annihilation operators we would get

$$\psi = \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_p}}\left(b_p e^{-ip.x} + c_p^+ e^{ip.x}\right)$$

$$\psi^* = \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_p}}\left(b_p^+ e^{+ip.x} + c_p e^{-ip.x}\right)$$

Our old fields had messy commutation with the charge

$$[Q, \phi^a(x)] = -i\epsilon^{ab}\phi^b(x)$$

But now

$$[Q, \psi] = -\psi$$

$$[Q, \psi^\dagger] = \psi^\dagger$$

ψ is an operator which lowers the charge by 1 either by annihilating a b-particle with charge $+1$ or creating a c-particle with charge -1. Likewise $\psi^\dagger$ always raises the charge either annihilating a c-particle of charge -1 or creating a b-particle with charge $+1$.

The commutators are interesting

$$[\psi(x,t), \psi(y,t)] = [\psi^\dagger(x,t), \psi^\dagger(y,t)]$$

$$= [\psi(x,t), \psi^\dagger(y,t)] = 0$$

But $\quad [\dot{\psi}^*(x,t), \dot{\psi}(y,t)] = 0$. However $[\psi^*(x,t), \dot{\psi}^*(y,t)] = 0$

This is also zero.

Indeed, the only non-zero equal time commutator are $\psi(x,t)$ and $\partial_0\psi^*(y,t)$ and

$$[\psi(x,t), \dot{\psi}^*(y,t)] = [\psi^*(x,t), \dot{\psi}^*(y,t)] = i\delta^{(3)}(\vec{x}-\vec{y})$$

Ofc if we try to write the Lagrangian in terms of ψ and ψ^* we shall get

$$\boxed{\mathcal{L} = \partial_\mu \psi^* \partial_\mu \psi - m^2 \psi^* \psi}$$

and EOM are

$$\boxed{\begin{aligned} \Box^2\psi + \mu^2\psi &= 0 \\ \Box^2\psi^* + \mu^2\psi^* &= 0 \end{aligned}}$$

Remember in the original basis we had $so(2)$ symmetry

$$\phi^1 \longrightarrow \phi^1\cos\lambda + \phi^2\sin\lambda$$

$$\phi^2 \longrightarrow \phi^2\cos\lambda - \phi^1\sin\lambda$$

In terms of new operators the Hamiltonian can be written as

$$H = \int d^3p \, \omega_p \left[b_p^\dagger b_p + c_p^\dagger c_p \right]$$

But infact the system has a larger invariance group of internal symmetries. including discrete internal symmetry. It has full $\underbrace{O(2)}$ invariance.

Meaning it is invariant not just under proper rotation but also under improper rotations ie reflections.

Lets choose

$$\phi^1 \longrightarrow \phi^1$$

$$\phi^2 \longrightarrow -\phi^2$$

so that in $\varphi - \varphi^*$ formalism

$$\varphi = \frac{1}{\sqrt{2}}(\phi^1 + i\phi^2) = \frac{1}{\sqrt{2}}(\phi^1 + i\phi^2)(\cos\lambda - i\sin\lambda)$$

$$= e^{-i\lambda}\varphi$$

$$\varphi^* = \frac{1}{\sqrt{2}}(\phi^1 - i\phi^2) = e^{i\lambda}\varphi^*$$

The group defined by this symmetry is $U(1)$. Unitary group in 1 dimension.

Anyway the symmetry that we are working now is

$$\phi^1 \longrightarrow \phi^1$$

$$\phi^2 \longrightarrow -\phi^2$$

Claim:– there is a unitary transformation that does this $\Rightarrow$

$$\left.\begin{aligned} \phi^1 &\longrightarrow U^+\phi^1 U = \phi^1 \\ \phi^2 &\longrightarrow U^+\phi^2 U = -\phi^2 \end{aligned}\right\}$$

$$a_p^{(1)} \to U^+ a_p^{(1)} U = a_p^{(1)}$$

$$a_p^{(2)} \to U^+ (a_p^{(2)}) U = -a_p^{(1)}$$

and so creation also.

Thus

$$\varphi = \frac{1}{\sqrt{2}}(\phi^1 + i\phi^2) \xrightarrow{U} \frac{1}{\sqrt{2}}(\phi^1 - i\phi^2) = \varphi^*$$

$$\text{Thus} \quad \underbrace{\varphi \xrightarrow{\mathcal{U}} \varphi^*}$$

You can say that $\mathcal{U}$ acting on any state turns all the b-type particles into c-type.

$$b_p = \frac{1}{\sqrt{2}}\left(a_p^{(1)} + i a_p^{(2)}\right) \xrightarrow{\mathcal{U}} \frac{1}{\sqrt{2}}\left(a_p^{(1)} - i a_p^{(2)}\right) = c_p$$

$$c_p \xrightarrow{\mathcal{U}} b_p$$

Such a transformation is called charge conjugation } We call this symmetry C

$$C : \begin{Bmatrix} b_p \\ c_p \end{Bmatrix} \longmapsto \mathcal{U}_c^{\dagger} \begin{Bmatrix} b_p \\ c_p \end{Bmatrix} \mathcal{U}_c = \begin{Bmatrix} c_p \\ b_p \end{Bmatrix}$$

Here we can see that applying the charge conjugation twice is just identity.

$$\mathcal{U}_c^2 = \mathbb{I} \qquad \text{and we have} \qquad \mathcal{U}^{\dagger} = \mathcal{U}^{-1} \text{ [unitary]}$$

$$\Rightarrow \quad \mathcal{U}_c = \mathcal{U}_c^{-1}$$

$$\Rightarrow \quad \boxed{\mathcal{U}_c = \mathcal{U}_c^{\dagger}} \Rightarrow$$ This means $\mathcal{U}_c$ is both unitary and

Hermitian.

Now we shall look at Parity :- Parity changes the sign of the spatial coordinates leaving time-coordinate untouched

$$P : \begin{cases} x \to -x \\ t \to t \end{cases}$$

Parity transformation is the same thing as a reflection followed by rotation about a normal to that plane by 180°. So a theory with rotational symmetry is parity invariant if and only if it is reflection-invariant.

But parity is improper rotation (its determinant is -1)

So until so long ago it was assumed that any realistic Physical theory would be parity-symmetric but Wu and her group discovered that parity was violated by β-decay.

Now an ordinary scalar (say m) is invariant under parity while an ordinary 3-vector like velocity changes sign

$$P: m \rightarrow m$$

$$P: v \rightarrow -v$$

on the other hand a cross product of two vector (Eg:- angular momentum

$L = r \times p$ picks two sign

and scalar triple product $w = a \cdot (b \times c)$ is scalar that changes sign

$$P: L \rightarrow L \qquad P: w \rightarrow -w$$

We call this "axial vectors" and "pseudo scalars" coz of such behaviours.

in field theory we can have scalar fields, vector fields, axial vector fields, pseudoscalar fields and so on.

Generally the action of parity is written as

$$P: \phi^a(x,t) \longrightarrow M^a_b \, \phi^b(-x,t) \rightarrow (2)$$

ie Parity forms a field at $(\bar{x},t)$ into some linear combination of fields at points $(-\vec{x},t)$. A theory is parity invariant if action is unchanged by (2).

Lets take an example:-

$$\mathcal{L}^{(1)} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\mu^2\phi^2 - \underbrace{g\phi^4}_{\text{interaction term}}$$

This Lagrangian has possess parity invariance

$$P: \phi(x,t) \longrightarrow \phi(-x,t) \quad (3)$$

scalar transformation law.

The Lagrangian changes by

$$\mathcal{L}(x,t) \rightarrow \mathcal{L}(-x,t)$$

But the action or EOM is unchanged.

This is implemented by the unitary operators

$$P: \begin{Bmatrix} a_p \\ a_p^\dagger \end{Bmatrix} \longrightarrow U_p^\dagger \begin{Bmatrix} a_p \\ a_p^\dagger \end{Bmatrix} U_p = \begin{Bmatrix} a_{-p} \\ a_{-p}^\dagger \end{Bmatrix}$$

Proof:→ Apply (3) to the defn of ϕ and change the integral variable $p \to -p$

$$\phi(x,t) \rightarrow \int \frac{d^3p}{(2\pi)^3} \left[a_p e^{-ip\cdot x} + a_p^{+} e^{ip\cdot x} \right]$$

$$\phi(x,t) \rightarrow \int \frac{d^3p}{(2\pi)^3} \left[a_p e^{-i\omega t + i\vec{p}\cdot\vec{x}} + a_p^{+} e^{i\omega t - i\vec{p}\cdot\vec{x}} \right]$$

$$\phi(-x,t) = \int \frac{d^3p}{(2\pi)^3} \left[a_p e^{-i\omega t - i\vec{p}\cdot\vec{x}} + a_p^{+} e^{i\omega t + i\vec{p}\cdot\vec{x}} \right]$$

$$= \int \frac{d^3p}{(2\pi)^3} \left[a_{-p} e^{-ip\cdot x} + a_{-p}^{+} e^{ip\cdot x} \right]$$

Thus parity takes the particle going $\longrightarrow$ and turns it in particle going $\longleftarrow$

Acting on the basis states

$$U_p \,|\vec{p}_1 \vec{p}_2, \dots \vec{p}_n\rangle = |-p_1, -p_2, \dots -p_n\rangle$$

Now there is an alternative parity transformation

$$P' : \phi(\vec{x},t) \longrightarrow -\phi(-\vec{x},t)$$. This transformation is also invariance of our Lagrangian

4

coz L is invariant under $\phi \to -\phi$ and product of symmetry is a symmetry.

pseudoscalar transformation law (one another way of defining parity)

The product CP is a symmetry but its just a matter of notation and one can always call (4) as a ty of parity or new def^n of parity

Consider a Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\mu^2\phi^2 - g\phi^4 - \underbrace{h\phi^3}$$

coz of this $\phi \to -\phi$ is no longer a good def^n of parity nor it is a symmetry. In this case the only sensible def^n of parity is scalar transformation law.

Finally lets look at time-reversal symmetry :-

Time reversal is rather peculiar coz unlike others it is not unitary represented by operators but by anti-unitary operators.

Consider a particle in 1D moving in potential. The classical theory is invariant under the time-reversal transformation

$$T : \begin{cases} q(t) \longrightarrow q(-t) \\ p(t) \longrightarrow -p(-t) \end{cases}$$

our first guess is that there should be unitary operator (lets call) U_T that effects this transformation:

$$U_T^{\dagger} \begin{Bmatrix} q(t) \\ p(t) \end{Bmatrix} U_T \overset{?}{=} \begin{Bmatrix} q(-t) \\ -p(-t) \end{Bmatrix}$$

However this leads to contradiction immediately. We know that

$$[q(t), p(t)] = i$$

Apply U_T to right and $U_T^{\dagger}$ to left

$$U_T^\dagger q(t)p(t) U_T - U_T^\dagger p(t) q(t) U_T = U_T^\dagger i U_T$$

$$\Rightarrow q(-t)[-p(-t)] + p(-t)q(-t) = U_T^\dagger i U_T = i$$

$$\Rightarrow -[q(-t), p(-t)] = i$$

commutation changes which shouldnot

Thus our hypothesis should be wrong. There is no "unitary operator".

(2) There is 2nd Contradiction also. We expect U_T if exists should reverse time evolution

$$U_T^\dagger e^{-iHt} U_T = e^{iHt} \quad [\text{expectation}]$$

Take d/dt on both side. at $t=0$ we get

$$U_T^\dagger (-iH) U_T = iH$$

Canceling i's give

$$\boxed{U_T^\dagger H U_T = -H}$$

H and $-H$ are related by unitary transformation. If H is bounded from below then it doesnot make sense that time translation will make the Hamiltonian negative.

The resolution here is to use anti-unitary operators.

Review of Operators :-

Defn :- An operator is unitary if two conditions are met

→ U is invertible.

→ for any two vectors a and b in Hilbert space

$$(Ua, Ub) = (a, b)$$ ie U preserves the norm.

Defn :- An operator U is linear if for 2 complex α and β and $a, b \in$ Hilbert space.

$$U(\alpha a + \beta b) = \alpha Ua + \beta Ub$$

This is sufficient to show that U is linear. But first

Defn :- The adjoint A^{+} of a linear operator is defined by

$$(a, A^{+} b) = (Aa, b)$$

Now if U is unitary then

$$(a, U^{-1} b) = (Ua, UU^{-1} b)$$

$$= (Ua, b) = (a, U^{+} b)$$

$$\Rightarrow \boxed{U^{-1} = U^{+}} \Rightarrow \text{If } U \text{ is unitary.}$$

The transformation of state $a \to Ua$ can be thought of as

$$(a, Ab) \to (Ua, AUb) = (a, U^{+} A U b)$$

$$\Rightarrow \boxed{A \to U^{+} A U}$$

Def^n:- An anti-unitary operator is an invertible operator given by Ω

$$(\Omega a, \Omega b) = (a, b)^{*} = (b, a)$$

One example of such operators is complex conjugation & of Schrodinger wave eqn

$$K(\alpha\psi_1 + \beta\psi_2) = \alpha^*\psi_1^* + \beta^*\psi_2^*$$

Likewise $(K\psi_1, K\psi_2) = (\psi_1^*, \psi_2^*) = (\psi_2, \psi_1) = (\psi_1, \psi_2)^*$

A fact about anti-unitary Ω is that Ω can be written as $\underbrace{UK}$
product of these.
↳ It can be proved.

Def^n:- An operator A is called anti-linear if

$$A(\alpha a + \beta b) = \alpha^* Aa + \beta^* Ab$$

Claim:- Ω is anti-linear

$$\Omega(\alpha a + \beta b) = UK(\alpha a + \beta b) = U(\alpha^* a + \beta^* b)$$
$$= \alpha^* Ua + \beta^* Ub$$
$$= \alpha^* UKa + \beta^* UKb$$
$$= \alpha^* \Omega a + \beta^* \Omega b$$

Now how does operators transforms under Ω

Consider $(a, Aa) \longrightarrow (\Omega a, A\Omega a)$
$= (A\Omega a, \Omega a) = (\Omega\Omega^{-1}A\Omega a, \Omega a)$

✓ states transforms as

if A is Hermitian

$(\Omega a, \Omega b)$
$= (b, a)$

$\zeta = (a, \Omega^{-1} A \Omega a)$

Thus $\boxed{A \to \Omega^{-1} A \Omega}$

The transformation on state vector can be alternatively thought of as transformation of operators (Hermitian)

Now

$$\Omega_T^{-1} [q(0), p(0)] \Omega_T$$

$$= \quad - [q(0), p(0)] = \Omega_T^{-1} i \Omega_T = -i$$

Thus there is no contradiction.

For the 2nd Contradiction,

$$\Omega_T^{-1} (-iH) \Omega_T = iH$$

$$\Rightarrow \quad \Omega_T^{-1} (-i) \Omega_T \Omega_T^{-1} H \Omega_T = iH$$

$$\Rightarrow \quad i \Omega_T^{-1} H \Omega_T = iH \quad \Rightarrow \quad \boxed{\Omega_T^{-1} H \Omega_T = H}$$

H is invariant under time-reversal,

Let us come to QFT, we have the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\mu^2\phi^2$$

As before we can multiply time reversal operator by any internal symmetry and obtain an equally good time-reversal operator. Its actually better to work in PT coz Acting on x^μ PT multiplies all four comp by -1. This operation commutes with Lorentz group.

Now if I have a particle with momentum vector $\longmapsto$ parity will reverse it $\longleftarrow$ but time reversal will reverse it again from $\longleftarrow$ to $\longmapsto$. So, I expect PT to do nothing to the momentum p. Therefore the anti-unitary operator

$$\Omega_{PT}\,|\vec{p}_1, \vec{p}_2, \cdots p_n\rangle = |\vec{p}_1, \vec{p}_2, \cdots \vec{p}_n\rangle$$

But this donot imply $\Omega_{PT} = 1$

lets look at the operators a_p and $a_p^\dagger$. Since Ω_{PT} do nothing to p. so one can easily deduce that

$$\Omega_{PT}\, a_p = a_p\, \Omega_{PT}$$

$$\Rightarrow \quad a_p = \Omega_{PT}^{-1}\, a_p\, \Omega_{PT}$$

It sure looks like Ω_{PT} acts like 1. But what about the fields?

We know

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_p}} \left(a_p\, e^{\overset{\text{This is}}{\overbrace{-ip\cdot x}}} + a_p^\dagger e^{ip\cdot x} \right)$$

When I apply Ω_T^{-1} and Ω_T then p donot change, a_p donot change but $e^{ip\cdot x}$ gets complex conjugated. So

$$\Omega_{PT}^{-1}\, \phi(x)\, \Omega_{PT} = \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_p}} \left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x} \right)$$

$$= \phi(-x)$$

The operator Ω_{PT} is not acting like 1. It turns the field at space-time point x^μ into the field at space point

Unitary Operator for Charge Conjugation :-

We shall define

number operator N_2 of particle 2 where

$$N_2 = \int d^3p \; a_p^{(2)\dagger} a_p^{(2)}$$

Then

$$u \,|\, \bar{p}_1, \bar{p}_2, \ldots \bar{p}_n \rangle = (-1)^{N_2} \,|\, \bar{p}_1, \bar{p}_2, \ldots \bar{p}_N \rangle$$

Thus

$$\left\{ \begin{array}{l} u^\dagger \phi^1 u = (-1)^{N_2} \phi^1 (-1)^{N_2} = \phi^1 \quad \text{and} \\ u^\dagger \phi^2 u = (-1)^{N_2} \phi^2 (-1)^{N_2} = -\phi^2 \end{array} \right.$$

Lets talk about this. Here N_2 commutes with ϕ^1 and so goes through

For this equation ϕ^2 will create or annihilate a type 2 particle, hence change the number by 1 and so if the first N_2 will make $(-1)^{N_2}$ positive or negative, the 2nd N_2 on the left hand side will make $(-1)^{N_2}$ the other way giving a minus sign in total.